SECTION 6.1
EXERCISE 15

6.1.15 Find the projection of $[1, 2, 1]$ on the subspace $sp([3, 1, 2], [1, 0, 1])$ in $\mathbb{R}^3$.

Solution

Let $W = sp([3, 1, 2], [1, 0, 1])$. By Note 6.1.A we can create matrix A with these vectors as rows and then by finding the nullspace of A (that is, by solving $A\vec{x} = \vec{0}$) we can find a basis for $W^\perp$. So we take $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ and consider $A\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - 3R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \text{ or } \begin{array}{l} x_1 + x_3 = 0 \\ x_2 - x_3 = 0 \end{array} \text{ or } \begin{array}{l} x_1 = -x_3 \\ x_2 = x_3 \\ x_3 = x_3 \end{array}$$

or with $r = x_3$ as a free variable

$$\begin{array}{l} x_1 = -r \\ x_2 = r \\ x_3 = r \end{array} \quad \text{where } r \in \mathbb{R}.$$

So $\vec{x} = r\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ where $r \in \mathbb{R}$ and $W^\perp = sp([-1, 1, 1])$.

Hence a basis for $W^\perp$ is $\{[-1, 1, 1]\}$.

By Note 6.1.B, we need to write $[1, 2, 1]$ as a linear combination of $[3, 1, 2]$, $[1, 0, 1]$, and $[-1, 1, 1]$. This leads us to consider (see Note 1.3.A)

$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 3 & 1 & -1 & 1 \\ 2 & 1 & 1 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -4 & -5 \\ 0 & 1 & -1 & -3 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -4 & -5 \\ 0 & 0 & 3 & 2 \end{array}\right] \xrightarrow{R_3 \to R_3/3}$$

SECTION 6.1
EXERCISE 15 (continued)

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -4 & -5 \\ 0 & 0 & 1 & 2/3 \end{array}\right] \begin{array}{c} R_1 \to R_1 - R_3 \\ \sim \\ R_2 \to R_2 + 4R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4/3 \\ 0 & 1 & 0 & -7/3 \\ 0 & 0 & 1 & 2/3 \end{array}\right].$$

So

$$[1,2,1] = \tfrac{4}{3}[3,1,2] - \tfrac{7}{3}[1,0,1] + \tfrac{2}{3}[-1,1,1]$$

and by Note 6.1.B,

$$\vec{b}_W = \text{proj}_W(\vec{b}) = \tfrac{4}{3}[3,1,2] - \tfrac{7}{3}[1,0,1]$$

$$= \left[4 - \tfrac{7}{3}, \tfrac{4}{3} - 0, \tfrac{8}{3} - \tfrac{7}{3}\right] = \boxed{\left[\tfrac{5}{3}, \tfrac{4}{3}, \tfrac{1}{3}\right]}. \quad \square$$