SECTION 6.1
NUMBER 25

6.1.25 Let W be a subspace of R^n and let $\vec{b} \in \mathbb{R}^n$. Prove that there is one and only one vector $\vec{p} \in W$ such that $\vec{b} - \vec{p}$ is perpendicular to every vector in W.

<u>Proof</u>

By Theorem 6.1(4), "Properties of $W^\perp$," we know that $\vec{b} = \vec{b}_W + \vec{b}_{W^\perp}$, or $\vec{b} - \vec{b}_W = b_{W^\perp}$. So we can take $\vec{p} = \vec{b}_W \in W$ to get one such value of $\vec{p}$.

Now suppose $\vec{p}_1$ and $\vec{p}_2$ are two vectors in W such that $\vec{b} - \vec{p}_1$ and $\vec{b} - \vec{p}_2$ are perpendicular to every vector in W. We now show that $\vec{p}_1 = \vec{p}_2$ (though we could also use the uniqueness of $\vec{b}_{W^\perp}$ in Theorem 6.1 to establish the uniqueness of $\vec{p}$). Notice that for any $\vec{w} \in W$ we have

$$
\begin{aligned}
(\vec{p}_1 - \vec{p}_2) \cdot \vec{w} &= \left((\vec{b} - \vec{p}_2) - (\vec{b} - \vec{p}_2)\right) \cdot \vec{w} \\
&= (\vec{b} - p_2) \cdot \vec{w} - (\vec{b} - \vec{p}_2) \cdot \vec{w} \\
&= 0 + 0 = 0 \quad \text{since } \vec{b} - \vec{p}_1,\ \vec{b} - p_2 \in W^\perp.
\end{aligned}
$$

So $\vec{p}_1 - \vec{p}_2 \in W^\perp$. But since $\vec{p}_1, \vec{p}_2 \in W$ then $\vec{p}_1 - \vec{p}_2 \in W$ also, so $\vec{p}_1 - \vec{p}_2$ is perpendicular to itself so that

$\|\vec{p}_1 - \vec{p}_2\|^2 = \langle \vec{p}_1 - \vec{p}_2, \vec{p}_1 - \vec{p}_2 \rangle = 0$ so that $\vec{p}_1 - \vec{p}_2 = \vec{0}$, or $\vec{p}_1 = \vec{p}_2$. Therefore the choice of $\vec{p}$ is unique. ∎