SECTION 6.1

NUMBER 27

6.1.27 Subspaces U and W of $\mathbb{R}^n$ are orthogonal if $\vec{u}\cdot\vec{w}=0$ for all $\vec{u}\in U$ and $\vec{w}\in W$. Let U and W be orthogonal subspaces of $\mathbb{R}^n$, and let $\dim(U)=n-\dim(W)$. Prove that each subspace is the orthogonal complement of the other.

Proof

We show $W=U^\perp$; the argument that $U=W^\perp$ follows in exactly the same way. Now $U^\perp$ is, by Definition 6.1 "Orthogonal Complement," the set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in U:

$$U^\perp=\{\vec{v}\in\mathbb{R}^n \mid \vec{v}\cdot\vec{u}=0 \text{ for all } \vec{u}\in U\}.$$

By hypothesis, each $\vec{w}\in W$ is in $U^\perp$ and so $W\subset U^\perp$. By Theorem 6.1(2), "Properties of $W^\perp$," $\dim(U^\perp)=n-\dim(U)$. By hypothesis, $\dim(W)=n-\dim(U)$, so $\dim(W)=\dim(U^\perp)$. Since $W\subset U^\perp$ and $\dim(W)=\dim(U^\perp)$ then by Exercise 3.2.36, $W=U^\perp$. ■