SECTION 6.1

NUMBER 39

6.1.39 Let S and T be nonempty subsets of an inner-product space V with the property that every vector in S is orthogonal to every vector in T. Prove that the span of S and the span of T are orthogonal subspaces of V.

Proof

Let $\vec{s} \in sp(S)$ and $\vec{t} \in sp(T)$. Then for some $\vec{s}_1, \vec{s}_2, \ldots, \vec{s}_n \in S$ and $\vec{t}_1, \vec{t}_2, \ldots, t_m \in T$ we have scalars $r_1, r_2, \ldots, r_n \in \mathbb{R}$ and $r_1', r_2', \ldots, r_m' \in \mathbb{R}$ such that

$r_1\vec{s}_1 + r_2\vec{s}_2 + \cdots + r_n\vec{s}_n = S$ and

$r_1'\vec{t}_1 + r_2'\vec{t}_2 + \cdots + r_m'\vec{t}_m \in T$. So

$$\langle \vec{s}, \vec{t} \rangle = \langle r_1\vec{s}_1 + r_2\vec{s}_2 + \cdots + r_n\vec{s}_n, r_1'\vec{t}_1 + r_2'\vec{t}_2 + \cdots + r_m'\vec{t}_m \rangle$$

$$= r_1r_1'\langle \vec{s}_1, \vec{t}_1\rangle + r_1r_2'\langle \vec{s}_1, \vec{t}_2\rangle + \cdots + r_1r_m'\langle \vec{s}_1, \vec{t}_m\rangle$$

$$+ r_2r_1'\langle \vec{s}_2, \vec{t}_1\rangle + r_2r_2'\langle \vec{s}_2, \vec{t}_2\rangle + \cdots + r_2r_m'\langle \vec{s}_2, \vec{t}_m\rangle$$

$$+ \cdots + r_nr_1'\langle \vec{s}_n, \vec{t}_1\rangle + r_nr_2'\langle \vec{s}_n, \vec{t}_2\rangle + \cdots + r_nr_m'\langle \vec{s}_n, \vec{t}_m\rangle$$

$= 0$ since every vector in S is orthogonal to every vector in T and so $\langle \vec{s}_i, \vec{t}_j \rangle = 0$ for all i, j.

So every vector in $sp(S)$ is orthogonal to every vector in $sp(T)$. That is, $sp(S)$ and $sp(T)$ are orthogonal subspaces of V. (See Exercise 27 for the definition of "orthogonal subspaces" when $V = \mathbb{R}^n$.)

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