SECTION 6.2
NUMBER 21

6.2.21 Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by $[1,0,1,0]$, $[1,1,1,0]$, and $[1,-1,0,1]$.

__Solution__

Denote the given vectors as $\vec{a}_1, \vec{a}_2, \vec{a}_3$.

Let $\vec{v}_1 = \vec{a}_1 = [1,0,1,0]$. Next,

$$\vec{v}_2 = \vec{a}_2 - \mathrm{proj}_{sp(\vec{v}_1)}(\vec{a}_2) = \vec{a}_2 - \frac{\vec{a}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$$

$$= [1,1,1,0] - \frac{[1,1,1,0]\cdot[1,0,1,0]}{[1,0,1,0]\cdot[1,0,1,0]}[1,0,1,0]$$

$$= [1,1,1,0] - \frac{2}{2}[1,0,1,0] = [0,1,0,0],$$

and $\vec{v}_3 = \vec{a}_3 - \mathrm{proj}_{sp(\vec{v}_1,\vec{v}_2)}(\vec{a}_3)$

$$= \vec{a}_3 - \frac{\vec{a}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{a}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$$

$$= [1,-1,0,1] - \frac{[1,-1,0,1]\cdot[1,0,1,0]}{[1,0,1,0]\cdot[1,0,1,0]}[1,0,1,0]$$

$$- \frac{[1,-1,0,1]\cdot[0,1,0,0]}{[0,1,0,0]\cdot[0,1,0,0]}[0,1,0,0]$$

$$= [1,-1,0,1] - \frac{1}{2}[1,0,1,0] - \frac{(-1)}{1}[0,1,0,0]$$

$$= [1,-1,0,1] - [\tfrac{1}{2}, 0, \tfrac{1}{2}, 0] + [0,1,0,0]$$

$$= [\tfrac{1}{2}, 0, -\tfrac{1}{2}, 1].$$

Next,

$$\|\vec{v}_1\| = \|[1, 0, 1, 0]\| = \sqrt{2}$$

$$\|\vec{v}_2\| = \|[0, 1, 0, 0]\| = 1$$

$$\|\vec{v}_3\| = \left\|\left[\tfrac{1}{2}, 0, \tfrac{-1}{2}, 1\right]\right\| = \sqrt{\tfrac{1}{4} + \tfrac{1}{4} + 1} = \sqrt{\tfrac{3}{2}}.$$

So

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}}[1, 0, 1, 0] = \left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right]$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{(1)}[0, 1, 0, 0] = [0, 1, 0, 0]$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \frac{1}{\sqrt{3/2}}\left[\frac{1}{2}, 0, \frac{-1}{2}, 1\right]$$

$$= \left[\frac{\sqrt{2}}{2\sqrt{3}}, 0, \frac{-\sqrt{2}}{2\sqrt{3}}, \sqrt{\frac{2}{3}}\right].$$

So the orthonormal basis is

$$\{\vec{q}_1, \vec{q}_2, \vec{q}_3\} = \left\{\left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right], [0, 1, 0, 0], \left[\frac{\sqrt{2}}{2\sqrt{3}}, 0, \frac{-\sqrt{2}}{2\sqrt{3}}, \sqrt{\frac{2}{3}}\right]\right\}.$$

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