SECTION 6.2
NUMBER 29

6.2.29 Let A be an $n \times k$ matrix. Prove that the column vectors of A are orthonormal if and only if $A^T A = I$. NOTE: This inspires the definition of "orthogonal matrix" in the next section.

<u>Proof</u>

By Definition 1.8, "Matrix Multiplication," if $C = AB$ then c_{ij} is the dot product of the ith row of A and the jth column of B. So the (i,j) entry of $A^T A$ (a $k \times k$ matrix) is the dot product of the ith row of A^T and the jth column of B. But the ith row of A^T is the same as the ith column of A, so the (i,j) entry of $A^T A$ is the dot product of the ith column of A with the jth column of A. Now if $A^T A = I$, then with $\vec{c}_1, \vec{c}_2, \dots, \vec{c}_k$ as the columns of A, $\vec{c}_i \cdot \vec{c}_j$ is the (i,j) entry of I. That is,

$$\vec{c}_i \cdot \vec{c}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}.$$

Since $\vec{c}_i \cdot \vec{c}_i = |\vec{c}_i|^2$, then the columns of A form an orthonormal set. Conversely, if the columns of A form an orthonormal set then

$$\vec{c}_i \cdot \vec{c}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

and so $A^T A = I$. ∎