SECTION 6.2
NUMBER 33

6.2.33 Find an orthonormal basis for sp(sin x, cos x) for $0 \le x \le \pi$ if the inner product is defined by $\langle f, g \rangle = \int_0^{\pi} f(x)\, g(x)\, dx$.

Solution

We apply the Gram-Schmidt Process.

Let $f_1 = \vec{a}_1 = \sin x$. Let

$$f_2 = \vec{a}_2 - \frac{\langle \vec{a}_2, f_1 \rangle}{\langle f_1, f_1 \rangle} f_1$$

$$= \cos x - \left(\frac{\int_0^{\pi} \cos x \sin x \, dx}{\int_0^{\pi} \sin x \sin x \, dx} \right) \sin x$$

$$= \cos x - \left(\frac{\frac{1}{2} \sin^2 x \Big|_0^{\pi}}{\left(\frac{1}{2} x - \frac{1}{4} \sin(2x) \right) \Big|_0^{\pi}} \right) \sin x$$

recall $\sin^2(x) = (1 - \cos(2x))/2$ (see below)

$$= \cos x - \left(\frac{\frac{1}{2} \sin^2(\pi) - \frac{1}{2} \sin^2(0)}{\left(\frac{1}{2}\pi - \frac{1}{4} \sin(2\pi) \right) - \left(\frac{1}{2}(0) - \frac{1}{4} \sin(0) \right)} \right) \sin x$$

$$= \cos x - \frac{0}{\pi/2} \sin x = \cos x.$$

So we have the orthogonal basis $\{f_1, f_2\} = \{\sin x, \cos x\}$.
We just need to normalize these functions.
We have

$$\|f_1\| = \langle f_1, f_1 \rangle^{1/2} = \left\{ \int_0^{\pi} \sin^2(x)\, dx \right\}^{1/2}$$

SECTION 6.2

NUMBER 33 (continued)

$$= \left\{ \int_0^{\pi} \frac{1 - \cos(2x)}{2}\, dx \right\}^{1/2} \quad \text{since } \sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$= \left\{ \left(\frac{1}{2}x - \frac{1}{4}\sin(2x) \right) \Big|_0^{\pi} \right\}^{1/2}$$

$$= \left\{ \left(\frac{1}{2}(\pi) - \frac{1}{4}\sin(2\pi) \right) - \left(\frac{1}{2}(0) - \frac{1}{4}\sin(0) \right) \right\}^{1/2} = \sqrt{\frac{\pi}{2}}.$$

Next,

$$\|f_2\| = \langle f_2, f_2 \rangle^{1/2} = \left\{ \int_0^{\pi} \cos^2(x)\, dx \right\}^{1/2}$$

$$= \left\{ \int_0^{\pi} \frac{1 + \cos(2x)}{2}\, dx \right\}^{1/2} \quad \text{since } \cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$= \left\{ \left(\frac{1}{2}x + \frac{1}{4}\sin(2x) \right) \Big|_0^{\pi} \right\}^{1/2}$$

$$= \left\{ \left(\frac{1}{2}\pi + \frac{1}{4}\sin(2\pi) \right) - \left(\frac{1}{2}(0) - \frac{1}{4}\sin(0) \right) \right\}^{1/2} = \sqrt{\frac{\pi}{2}}.$$

So an orthonormal basis is

$$\{ f_1/\|f_1\|, f_2/\|f_2\| \} = \boxed{\left\{ \sqrt{\frac{2}{\pi}}\sin x, \sqrt{\frac{2}{\pi}}\cos x \right\}}. \quad \square$$