SECTION 6.2
NUMBER 35

6.2.35 Find an orthonormal basis for sp$(1, e^x)$ for $0 \le x \le 1$ if the inner product is defined by $\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx$.

Solution

We apply the Gram-Schmidt Process.

Let $f_1 = \vec{a}_1 = 1$. Let

$$f_2 = \vec{a}_2 - \frac{\langle \vec{a}_2, f_1 \rangle}{\langle f_1, f_1 \rangle} f_1$$

$$= e^x - \left(\frac{\int_0^1 1 \cdot e^x\, dx}{\int_0^1 1\, dx} \right) 1 = e^x - \left(\frac{e^x \Big|_0^1}{x \Big|_0^1} \right) 1$$

$$= e^x - \left(\frac{e^1 - e^0}{1 - 0} \right) 1 = e^x - e + 1.$$

So we have the orthogonal basis $\{f_1, f_2\} = \{1, e^x - e + 1\}$.

We just need to normalize these functions.

We have

$$\|f_1\| = \langle f_1, f_1 \rangle^{1/2} = \left\{ \int_0^1 1\, dx \right\}^{1/2} = \left\{ x \Big|_0^1 \right\}^{1/2}$$

$$= \{1 - 0\}^{1/2} = 1.$$

Next

$$\|f_2\| = \langle f_2, f_2 \rangle^{1/2} = \left\{ \int_0^1 (e^x - e + 1)(e^x - e + 1)\, dx \right\}^{1/2}$$

$$= \left\{ \int_0^1 \left(e^{2x} - e^{x+1} + e^x - e^{x+1} + e^2 - e + e^x - e + 1\right) dx \right\}^{1/2}$$

$$= \left\{ \int_0^1 \left(e^{2x} - 2e^{x+1} + 2e^x + e^2 - 2e + 1\right) dx \right\}^{1/2}$$

$$= \left\{ \left(\frac{1}{2} e^{2x} - 2e^{x+1} + 2e^x + (e^2 - 2e + 1)x \right) \Big|_0^1 \right\}^{1/2}$$

SECTION 6.2
NUMBER 35 (continued)

$$= \left\{ \left(\tfrac{1}{2} e^{2(1)} - 2e^{(1)+1} + 2e^{(1)} + (e^2 - 2e + 1)(1) \right) - \left(\tfrac{1}{2} e^{2(0)} - 2e^{(0)+1} + 2e^{(0)} + (e^2 - 2e + 1)(0) \right) \right\}^{1/2}$$

$$= \left\{ \tfrac{1}{2} e^2 - 2e^2 + 2e + e^2 - 2e + 1 - \tfrac{1}{2} + 2e - 2 \right\}^{1/2}$$

$$= \left(-\tfrac{1}{2} e^2 + 2e - \tfrac{3}{2} \right)^{1/2} = \sqrt{\frac{-e^2 + 4e - 3}{2}}.$$

So an orthonormal basis is

$$\left\{ \frac{f_1}{\|f_1\|}, \frac{f_2}{\|f_2\|} \right\} = \boxed{\left\{ 1, \sqrt{\frac{2}{-e^2 + 4e - 3}} \, (e^x - e + 1) \right\}}. \quad \square$$