

SECTION 6.2
NUMBER 9

(1)

6.2.9 Transform the basis $\{[1, 0, 1], [0, 1, 2], [2, 1, 0]\}$ for \mathbb{R}^3 into an orthonormal basis using the Gram-Schmidt process.

Proof

Denote the given vectors as $\vec{a}_1, \vec{a}_2, \vec{a}_3$.

Let $\vec{v}_1 = \vec{a}_1 = [1, 0, 1]$. Next,

$$\vec{v}_2 = \vec{a}_2 - \text{proj}_{\vec{v}_1}(\vec{a}_2) = \vec{a}_2 - \frac{\vec{a}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$= [0, 1, 2] - \frac{[0, 1, 2] \cdot [1, 0, 1]}{[1, 0, 1] \cdot [1, 0, 1]} [1, 0, 1]$$

$$= [0, 1, 2] - \frac{2}{2} [1, 0, 1] = [-1, 1, 1], \text{ and}$$

$$\vec{v}_3 = \vec{a}_3 - \text{proj}_{\text{span}(\vec{v}_1, \vec{v}_2)}(\vec{a}_3)$$

$$= \vec{a}_3 - \frac{\vec{a}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{a}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$= [2, 1, 0] - \frac{[2, 1, 0] \cdot [1, 0, 1]}{[1, 0, 1] \cdot [1, 0, 1]} [1, 0, 1] -$$

$$- \frac{[2, 1, 0] \cdot [-1, 1, 1]}{[-1, 1, 1] \cdot [-1, 1, 1]} [-1, 1, 1]$$

$$= [2, 1, 0] - \frac{2}{2} [1, 0, 1] - \frac{(-1)}{3} [-1, 1, 1]$$

$$= \left[\frac{2}{3}, \frac{4}{3}, -\frac{2}{3} \right].$$

Finally, $\|\vec{v}_1\| = \|[1, 0, 1]\| = \sqrt{2}$,

$$\|\vec{v}_2\| = \|[-1, 1, 1]\| = \sqrt{3}, \text{ and}$$

$$\|\vec{v}_3\| = \left\| \left[\frac{2}{3}, \frac{4}{3}, \frac{-2}{3} \right] \right\| = \sqrt{\frac{24}{9}} = \frac{2}{3}\sqrt{6}$$

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{[1, 0, 1]}{\sqrt{2}} = \left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right]$$

$$\vec{q}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{[-1, 1, 1]}{\sqrt{3}} = \left[\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right], \text{ and}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \frac{\left[\frac{2}{3}, \frac{4}{3}, \frac{-2}{3} \right]}{\frac{2}{3}\sqrt{6}} = \left[\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right].$$

An orthonormal basis is given by

$$\{\vec{q}_1, \vec{q}_2, \vec{q}_3\} = \left\{ \left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right], \left[\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right], \left[\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right] \right\}.$$

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