SECTION 6.3
NUMBER 25

6.3.25 Let A be an $n \times n$ matrix such that $A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y}$ for all vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$. Prove that A is an orthogonal matrix.

<u>Proof</u>

Let $\hat{e}_i$ be the ith standard basis vector of $\mathbb{R}^n$. Then, by Note 2.3.A, $A\hat{e}_i$ is the ith column of A. So the dot product of the ith column of A with the jth column of A is $A\hat{e}_i \cdot A\hat{e}_j$. By hypothesis

$$A\hat{e}_i \cdot A\hat{e}_j = \hat{e}_i \cdot \hat{e}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}.$$

So the columns of A are orthonormal. These columns form a basis for their span by Theorem 6.2, "Orthogonal Bases," and since there are n columns in A then the span is an n-dimensional subspace of $\mathbb{R}^n$. By Exercise 3.2.36 the span is all of $\mathbb{R}^n$. So by Theorem 6.5, "Characterizing Properties of an Orthogonal Matrix" (the (2) implies (3) part), A is an orthogonal matrix. ∎