SECTION 6.3
NUMBER 27

6.3.27 Prove that the real eigenvalues of an orthogonal matrix must be equal to 1 or -1.

Proof

Let A be an orthogonal (square) matrix with eigenvalue λ and corresponding eigenvector $\vec{v}$. That is, suppose $A\vec{v} = \lambda\vec{v}$. By Theorem 6.6, "Preservation of Length," $\|A\vec{v}\| = \|\vec{v}\|$. So $\|\vec{v}\| = \|A\vec{v}\| = \|\lambda\vec{v}\| = |\lambda|\,\|\vec{v}\|$ (by Theorem 1.2(2), "Homogeneity of Norm"). Since $\vec{v}$ is not the zero vector by the definition of eigenvector (Definition 5.2), then $|\lambda| = 1$ and so $\lambda = \pm 1$. ∎