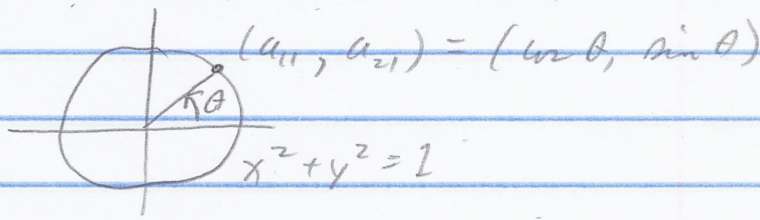
SECTION 6.3
NUMBER 39

6.3.39 Show that every 2×2 orthogonal matrix is of one of two forms; either

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \text{ or } \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ for some angle } \theta.$$

Solution

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a 2×2 orthogonal matrix. Then by Theorem 6.5, "Characterizing Properties of an Orthogonal Matrix" (the (3) implies (2) part), the vector $\vec{c}_1 = [a_{11} \; a_{21}]^T$ is a unit vector and so $(a_{11})^2 + (a_{21})^2 = 1$. So the point (a_{11}, a_{21}) in the Cartesian plane lies on the graph of $x^2+y^2=1$. So we can take $a_{11} = \cos(\theta)$ and $a_{21} = \sin(\theta)$ for some $\theta \in \mathbb{R}$ (or $\theta \in [0, 2\pi)$; the point (a_{11}, a_{12}) is on the intersection of the circle $x^2+y^2=1$ and the terminal side of angle θ in standard position):



Also by Theorem 6.5 (the (3) implies (2) part again), the vector $\vec{c}_2 = [a_{12} \; a_{22}]^T$ is a unit vector and is perpendicular to $\vec{c}_1$. So $(a_{12})^2 + (a_{22})^2 = 1$ and, as above, $a_{12} = \cos(\varphi)$ and $a_{22} = \sin(\varphi)$ for some $\varphi \in \mathbb{R}$.

SECTION 6.3
NUMBER 39 (continued)

But since $\vec{c}_1$ and $\vec{c}_2$ are perpendicular then we can take $\varphi = \theta + \pi/2$ or $\varphi = \theta - \pi/2$.
Recall that

$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$ and
$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$. So

$$\sin(\theta \pm \pi/2) = \sin\theta \cos(\pi/2) \pm \cos(\theta) \sin(\pi/2)$$
$$= 0 \sin\theta \pm 1 \cos\theta = \pm \cos\theta \text{ and}$$
$$\cos(\theta \pm \pi/2) = \cos\theta \cos(\pi/2) \mp \sin\theta \sin(\pi/2)$$
$$= 0 \cos\theta \mp 1 \sin\theta = \mp \sin\theta.$$

So either $a_{12} = -\sin\theta$ and $a_{22} = \cos\theta$ or
$a_{12} = \sin\theta$ and $a_{22} = -\cos\theta$.

So either

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ or}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}, \text{ as claimed. } \square$$