SECTION 6.3
NUMBER 41

6.3.41 (Real Householder Matrix) Let $\vec{v}$ be a nonzero column vector in $\mathbb{R}^n$. Prove that

$$C_v = \mathcal{I} - \frac{2}{\vec{v}\cdot\vec{v}}(\vec{v}\,\vec{v}^T)$$ is an orthogonal matrix.

Proof

We use Definition 6.4, "Orthogonal Matrix."

We have

$$C_v^T C_v = \left(\mathcal{I} - \frac{2}{\vec{v}\cdot\vec{v}}(\vec{v}\,\vec{v}^T)\right)^T\left(\mathcal{I} - \frac{2}{\vec{v}\cdot\vec{v}}(\vec{v}\,\vec{v}^T)\right)$$

$$= \left(\mathcal{I}^T - \frac{2}{\vec{v}\cdot\vec{v}}(\vec{v}\,\vec{v}^T)\right)\left(\mathcal{I} - \frac{2}{\vec{v}\cdot\vec{v}}(\vec{v}\,\vec{v}^T)\right)$$

by Note 1.3.B

$$= \mathcal{I} - \frac{2}{\vec{v}\cdot\vec{v}}(\vec{v}\,\vec{v}^T) - \frac{2}{\vec{v}\cdot\vec{v}}(\vec{v}\,\vec{v}^T) + \left(\frac{2}{\vec{v}\cdot\vec{v}}\right)^2(\vec{v}\,\vec{v}^T\vec{v}\,\vec{v}^T)$$

$$= \mathcal{I} - \frac{4}{\vec{v}\cdot\vec{v}}(\vec{v}\,\vec{v}^T) + \left(\frac{2}{\vec{v}\cdot\vec{v}}\right)^2(\vec{v}\cdot\vec{v})(\vec{v}\,\vec{v}^T)$$

since $\vec{v}^T\vec{v} = \vec{v}\cdot\vec{v}$

$$= \mathcal{I} - \frac{4}{\vec{v}\cdot\vec{v}}(\vec{v}\,\vec{v}^T) + \frac{4}{\vec{v}\cdot\vec{v}}(\vec{v}\,\vec{v}^T) = \mathcal{I}.$$

So C_v is an orthogonal matrix. ∎