Chapter 1. Vectors, Matrices, and Linear Spaces

1.2. The Norm and Dot Product

Note. In the previous section we mentioned that in physics a vector is an object with magnitude and direction. In this section we define the magnitude of a vector and give a operation, the dot product, which will let us compute both the magnitude of a vector and to define the angle between two vectors.

Definition 1.5. Let $\vec{v} = [v_1, v_2, \ldots, v_n] \in \mathbb{R}^n$. The *norm* or *magnitude* of $\vec{v}$ is

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{\sum_{\ell=1}^{n} (v_\ell)^2}.$$

Note. Definition 1.5 is consistent with the idea of a vector $\vec{v}$ in $\mathbb{R}^n$, say $\vec{v} = [v_1, v_2, \ldots, v_n]$, as an arrow (in standard position) with its tail at the origin $(0, 0, \ldots, 0)$ and its head at the point $(v_1, v_2, \ldots, v_n)$ in an n-dimensional Cartesian coordinate system. This distance between points $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ is $d = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots (x_n - y_n)^2}$ (this is based on the Pythagorean Theorem). So the length of the arrow representing $\vec{v}$ has length

$$\sqrt{(v_1 - 0)^2 + (v_2 - 0)^2 + \cdots + (v_n - 0)^2} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \|\vec{v}\|$$

(see also Exercise 1.2.31). We now state some properties of the norm function.

Theorem 1.2. Properties of the Norm in $\mathbb{R}^n$.

For all $\vec{v}, \vec{w} \in \mathbb{R}^n$ and for all scalars $r \in \mathbb{R}$, we have:

1. $\|\vec{v}\| \geq 0$ and $\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$.

2. $\|r\vec{v}\| = |r|\|\vec{v}\|$.

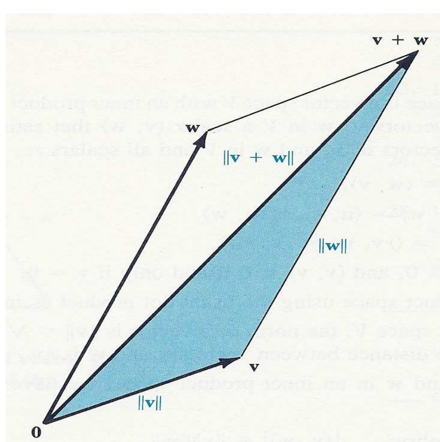
3. $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ (the Triangle Inequality).

Proof of (2). Let $\vec{v} \in \mathbb{R}^n$ and let r be a scalar in $\mathbb{R}$. Since $\vec{v} \in \mathbb{R}^n$, by Definition 1.A, "Vectors in $\mathbb{R}^n$," we have that $\vec{v} = [v_1, v_2, \ldots, v_n]$. We have

$$\begin{aligned} \|r\vec{v}\| &= \|r[v_1, v_2, \ldots, v_n]\| \\ &= \|[rv_1, rv_2, \ldots, rv_n]\| \text{ by Definition 1.1(3), "Scalar Multiplication"} \\ &= \sqrt{(rv_1)^2 + (rv_2)^2 + \cdots + (rv_n)^2} \text{ by Definition 1.5} \\ &= \sqrt{r^2(v_1^2 + v_2^2 + \cdots + v_n^2)} = \sqrt{r^2}\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = |r|\|\vec{v}\|. \qquad \blacksquare \end{aligned}$$

Note. The proof of Theorem 1.2(1) is easy; we will prove Theorem 1.2(3) later in this section.

Note. A picture for the Triangle Inequality is:



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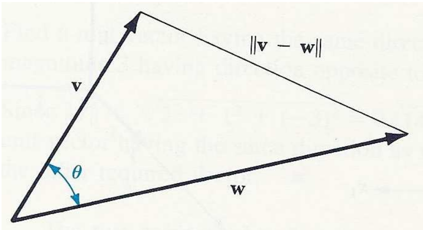
Definition. A vector with norm 1 is called a *unit vector*. When writing, unit vectors are frequently denoted with a "hat": $\hat{\imath}$.

Example. Page 31 number 8.

Definition 1.6. The *dot product* for $\vec{v} = [v_1, v_2, \ldots, v_n]$ and $\vec{w} = [w_1, w_2, \ldots, w_n]$ is

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \sum_{\ell=1}^{n} v_\ell w_\ell.$$

Notice. If we let θ be the angle between nonzero vectors $\vec{v}$ and $\vec{w}$, then we get by the Law of Cosines:



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$$\|\vec{v}\|^2 + \|\vec{w}\|^2 = \|\vec{v} - \vec{w}\|^2 + 2\|\vec{v}\|\|\vec{w}\| \cos\theta$$

or

$$(v_1^2 + v_2^2 + \cdots + v_n^2) + (w_1^2 + w_2^2 + \cdots + w_n^2)$$
$$= (v_1 - w_1)^2 + (v_2 - w_2)^2 + \cdots + (v_n - w_n)^2 + 2\|\vec{v}\|\|\vec{w}\| \cos\theta$$

or $2v_1w_1 + 2v_2w_2 + \cdots + 2v_nw_n = 2\|\vec{v}\|\|\vec{w}\| \cos\theta$ or $2\vec{v} \cdot \vec{w} = 2\|\vec{v}\|\|\vec{w}\| \cos\theta$ or $\cos\theta = \dfrac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}$.

Definition. The *angle* between nonzero vectors $\vec{v}$ and $\vec{w}$ is

$$\cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right) = \arccos\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right).$$

Example. Page 31 number 12.

Theorem 1.3. Properties of Dot Products.

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and let $r \in \mathbb{R}$ be a scalar. Then

D1. Commutivity of $\cdot$: $\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$.

D2. Distribution of $\cdot$ over Vector Addition: $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$.

D3. $r(\vec{v}\cdot\vec{w}) = (r\vec{v})\cdot\vec{w} = \vec{v}\cdot(r\vec{w})$.

D4. $\vec{v}\cdot\vec{v} \geq 0$ and $\vec{v}\cdot\vec{v} = 0$ if and only if $\vec{v} = \vec{0}$.

Example. Page 33 number 42b (Prove D2).

Note 1.2.A. $\|\vec{v}\|^2 = \vec{v}\cdot\vec{v}$.

Definition. Two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ are *perpendicular* or *orthogonal*, denoted $\vec{v} \perp \vec{w}$, if $\vec{v}\cdot\vec{w} = 0$.

Examples. Page 31 numbers 14 and 16.

Page 26 Example 7. Prove that the sum of the squares of the lengths of the diagonals of a parallelogram in $\mathbb{R}^n$ is equal to the sum of the squares of the lengths of the sides. This is the *parallelogram relation* or the *parallelogram law.*

Note. We saw above that ASSUMING vectors in $\mathbb{R}^n$ satisfy Figure 1.2.24 (see the proof of Example 7) implies $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\|\vec{w}\|}$. We know that for any θ we have $-1 \leq \cos\theta \leq 1$ and so this gives

$$-\|\vec{v}\|\|\vec{w}\| \leq \vec{v}\cdot\vec{w} \leq \|\vec{v}\|\|\vec{w}\| \text{ or } |\vec{v}\cdot\vec{w}| \leq \|\vec{v}\|\|\vec{w}\|.$$

Though a valid argument, we now give an algebraic proof of this inequality.

Theorem 1.4. Schwarz's Inequality.

Let $\vec{v},\vec{w} \in \mathbb{R}^n$. Then

$$|\vec{v}\cdot\vec{w}| \leq \|\vec{v}\|\|\vec{w}\|.$$

Note. The whole purpose of introducing Schwarz's Inequality is to prove Theorem 1.2(3), "The Triangle Inequality."

Theorem 1.5. The Triangle Inequality.

Let $\vec{v}, \vec{w} \in \mathbb{R}^n$. Then $\|\vec{v} + \vec{w}\| \le \|\vec{v}\| + \|\vec{w}\|$.

Proof.

$$\begin{aligned} \|\vec{v} + \vec{w}\|^2 &= (\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) \text{ by Note 1.2.A} \\ &= \vec{v} \cdot \vec{v} + 2\vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w} \text{ by Theorem 1.3(D1) and (D2),} \\ &\quad \text{“Commutivity and Distribution of Dot Product”} \\ &\le \|\vec{v}\|^2 + 2\|\vec{v}\|\|\vec{w}\| + \|\vec{w}\|^2 \text{ by Schwarz's Inequality} \\ &= (\|\vec{v}\| + \|\vec{w}\|)^2. \end{aligned}$$

Taking square roots yields $\|\vec{v} + \vec{w}\| \le \|\vec{v}\| + \|\vec{w}\|$. ∎

Example. Page 31 number 36.

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