Chapter 4. Determinants

4.2 The Determinant of a Square Matrix

Note. In this section we define the determinant of an $n \times n$ matrix. We will do so recursively by defining the determinant of $n \times n$ matrix A in terms of related $(n-1) \times (n-1)$ "submatrices" of A. This was foreshadowed in the previous section when we defined the determinant of a 3×3 matrix in terms of three 2×2 matrices.

Definition. The *minor matrix* A_{ij} of an $n \times n$ matrix A is the $(n-1) \times (n-1)$ matrix obtained from A by eliminating the ith row and the jth column of A.

Example 4.2.A. Find A_{11}, A_{12}, and A_{13} for

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Definition 4.1a. The determinant of A_{ij} times $(-1)^{i+j}$ is the *cofactor* of entry a_{ij} in A, denoted a'_{ij}.

Example. Page 262 Number 12.

Note. We can write determinants of 3×3 matrices in terms of cofactors:

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}|A_{11}| - a_{12}|A_{12}| + a_{13}|A_{13}|$$

$$= a_{11}a'_{11} + a_{12}a'_{12} + a_{13}a'_{13}.$$

Note. The following definition is *recursive*. For example, in order to process the definition for $n = 4$ you must process the definition for $n = 3$, $n = 2$, and $n = 1$.

Definition 4.1b. The *determinant* of a 1×1 matrix is its single entry. Let $n > 1$ and assume the determinants of order less than n have been defined. Let $A = [a_{ij}]$ be an $n \times n$ matrix. The *cofactor* of a_{ij} in A is $a'_{ij} = (-1)^{i+j}\det(A_{ij})$ (as defined above). The *determinant* of A is

$$\det(A) = a_{11}a'_{11} + a_{12}a'_{12} + \cdots + a_{1n}a'_{1n} = \sum_{i=1}^{n} a_{1i}a'_{1i}.$$

Example 4.2.B. Find the determinant of $A = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 2 \\ 4 & 0 & 1 & 4 \\ 1 & 0 & 2 & 1 \end{bmatrix}$.

Note. The following result can be useful; in particular, it allows us to take advantage of zeros in a matrix to simplify computations.

Theorem 4.2. General Expansion by Minors.

Let A be an $n \times n$ matrix. Then for any $1 \leq r \leq n$ and $1 \leq s \leq n$ we have that the determinant of A is

$$\det(A) = a_{r1}a'_{r1} + a_{r2}a'_{r2} + \cdots + a_{rn}a'_{rn} \qquad (4)$$

and

$$\det(A) = a_{1s}a'_{1s} + a_{2s}a'_{2s} + \cdots + a_{ns}a'_{ns} \qquad (5)$$

where a'_{ij} is the cofactor of A_{ij} given in Definition 4.1.

Note. A proof of Theorem 4.2 is given in Appendix B (see pages A-7–A-9). It is a lengthy proof and requires a knowledge of mathematical induction (which is given in Appendix A).

Definition. Equation (4) is the *expansion of det(A) by minors on the rth row* of A, and equation (5) is the *expansion of det(A) by minors on the sth column* of A.

Example 4.3.C. Find the determinant of $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 4 & 5 & 9 \\ 1 & 15 & 6 & 57 \end{bmatrix}$.

Example. Page 255 Example 4. Show that the determinant of an upper- or lower-triangular square matrix is the product of its diagonal elements.

Note. We combine the properties of determinants in the following theorem.

Theorem 4.2.A. Properties of the Determinant.

Let A be a square matrix.

1. The Transpose Property: $\det(A) = \det(A^T)$.

2. The Row-Interchange Property: If two different rows of a square matrix A are interchanged, the determinant of the resulting matrix is $-\det(A)$.

3. The Equal-Rows Property: If two rows of a square matrix A are equal, then $\det(A) = 0$.

4. The Scalar-Multiplication Property: If a single row of a square matrix A is multiplied by a scalar r, the determinant of the resulting matrix if $r\det(A)$.

5. The Row-Addition Property: If the product of one row of A by a scalar r is added to a different row of A, the determinant of the resulting matrix is the same as $\det(A)$.

Note 4.2.A. The properties of Theorem 4.2.A imply that we can perform elementary row operations on a matrix to introduce entries of 0 and simplify computations of determinants (though we have to track how the row operations affect the determinant). Since the computations of determinants is "computationally intense," it is worth the trouble to perform the row reduction. In fact, if $A \sim B$ and B results from a sequence of elementary row operations which do not involve multiplication of a row by a scalar, then $\det(A) = \pm\det(B)$. (we only need to know the number of row interchanges in converting A into B). In fact, we can always row reduce any A to a row echelon form without multiplying a row by scalars. We only must multiply by a scalar if we want the pivots to be 1. These observations are useful in the proofs of the following two results.

Example. Page 261 Number 8.

Note. We now add one more result to our list of conditions which are equivalent to the invertibility of a matrix (see also Theorem 1.12 of Section 1.5, Theorem 1.16 of Section 1.6, and Theorem 2.6 of Section 2.2).

Theorem 4.3. Determinant Criterion for Invertibility.
A square matrix A is invertible if and only if $\det(A) \neq 0$. Equivalently, A is singular if and only if $\det(A) = 0$.

Note. We state on last property of determinants which we will occasionally use.

Theorem 4.4. The Multiplicative Property.
If A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$.

Examples. Page 262 Number 28, Page 262 Number 30, Page 262 Number 32.

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