Chapter 5. Eigenvalues and Eigenvectors

5.1 Eigenvalues and Eigenvectors

Note. In optional Section 1.7, "Application to Population Distribution," a model is introduced in which a population is partitioned into several "states." The percentage of the population in state i is denoted p_i and for n states this results in population distribution vector $\vec{p} = [p_1, p_2, \ldots, p_n]$. A transition matrix T then describes how the population changes from one generation to the next (see Exercises 34-39 in Section 1.7). So after n time steps, the population distribution vector has become $T^n\vec{p}$. So this is one reason one might be interested in raising a matrix to a power. In this chapter we give a shortcut to the process of raising a matrix to a power.

Example. Page 286 Example 1. Fraleigh and Beauregard introduce a classic example to motivate the study of eigenvalues and eigenvectors. Suppose a pair of newly born rabbits are introduced to an environment. Suppose each pair of rabbits produces no offspring during the first month of their lives but that each pair produces one new pair each subsequent month. Start with $F_1 = 1$ newly born pair in the first month. In the second month there are also $F_2 = 1$ pair of rabbits, since the original pair is not old enough to reproduce. In the third month there are $F_3 = 2$ pairs, including one newly born pair. In the fourth month there are $F_4 = 2 + 1 = 3$ pairs since the original pair produces a new pair. Now the first new pair starts to reproduce and we have $F_5 = F_4 + F_3 = 3 + 2 = 5$. Under this model (regardless of biological realism), at the kth month the number of pairs

F_k is the number present in the previous month F_{k-1} (since the model does not account for mortality) plus the number present two months earlier, F_{k-2}, since this is the number of pairs of breeding age and so each of these produces a new pair. That is, $F_k = F_{k-1} + F_{k-2}$. For initial conditions we define $F_0 = 0$. We then get the *Fibonacci sequence*: $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$. If we represent the number of pairs in months $k-1$ and k as $\begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix}$ then we have $F_k = F_{k-1} + F_{k-2}$ so that $\begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k-1} \\ F_{k-2} \end{bmatrix} = \begin{bmatrix} F_{k-1} + F_{k-2} \\ F_{k-1} \end{bmatrix}$. Starting with $\begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we get

$$\begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k-1} \\ F_{k-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} F_{k-2} \\ F_{k-3} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^3 \begin{bmatrix} F_{k-3} \\ F_{k-4} \end{bmatrix}$$

$$= \cdots = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k-1} \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Notice that even though $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ consists only of 0's and 1's, it is still time consuming to calculate the powers of A. We'll use eigenvalues to find an explicit formula for $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k-1}$ in Section 5.3 and we'll see that (see page 319):

$$F_k = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right).$$

Note. We won't see how eigenvalues and eigenvectors are used in addressing these problems until the next section. So we now define these terms and show how to calculate them.

Definition 5.1. Let A be an $n \times n$ matrix. A scalar λ is an *eigenvalue* of A if there is a nonzero column vector $\vec{v} \in \mathbb{R}^n$ such that $A\vec{v} = \lambda\vec{v}$. The vector $\vec{v}$ is then an *eigenvector* of A corresponding to λ.

Note. If $A\vec{v} = \lambda\vec{v}$ then $A\vec{v} - \lambda\vec{v} = \vec{0}$ and so $(A - \lambda\mathcal{I})\vec{v} = \vec{0}$. This equation has a nontrivial solution only when $\det(A - \lambda\mathcal{I}) = 0$ by Theorem 4.3, "Determinant Criterion for Invertibility," and Theorem 1.16 (the contrapositive of (1)$\Rightarrow$(3), which would be stated as "If A is not invertible then the system $A\vec{x} = \vec{0}$ has multiple solutions").

Definition. $\det(A - \lambda\mathcal{I})$ is a polynomial of degree n with variable λ (where A is $n \times n$) called the *characteristic polynomial* of A, denoted $p(\lambda)$, and the equation $p(\lambda) = 0$ is called the *characteristic equation.*

Note 5.1.A. Since an eigenvector must be nonzero by definition, then the condition $\det(A - \lambda\mathcal{I}) = 0$ is necessary (and sufficient) for λ to be an eigenvalue. We then have that $(A - \lambda\mathcal{I})\vec{v} = \vec{0}$ for some nonzero $\vec{v}$ and so to find the eigenvectors of λ we solve the homogeneous system of equations $(A - \lambda\mathcal{I})\vec{v} = \vec{0}$ where the unknowns are the components of $\vec{v} = [v_1, v_2, \ldots, v_n]$. Since the system $(A - \lambda\mathcal{I})\vec{v} = \vec{0}$ has a nonzero solution $\vec{v}$, then any multiple of $\vec{v}$ is a solution and so there are an infinite number of eigenvectors associated with λ. This is reflected in the solving of the system $(A - \lambda\mathcal{I})\vec{v} = \vec{0}$ where we'll see that there is at least one free variable associated with eigenvalue λ.

Examples. Page 300 Number 8, Page 300 Number 14. See Page 292 Example 4, Page 293 Example 5, and Page 294 Example 6 for more worked examples.

Theorem 5.1. Properties of Eigenvalues and Eigenvectors.
Let A be an $n \times n$ matrix.

1. If λ is an eigenvalue of A with $\vec{v}$ as a corresponding eigenvector, then λ^k is an eigenvalue of A^k, again with $\vec{v}$ as a corresponding eigenvector, for any positive integer k.

2. If λ is an eigenvalue of an invertible matrix A with $\vec{v}$ as a corresponding eigenvector, then $\lambda \neq 0$ and $1/\lambda$ is an eigenvalue of A^{-1}, again with $\vec{v}$ as a corresponding eigenvector.

3. If λ is an eigenvalue of A, then the set E_λ consisting of the zero vector together with all eigenvectors of A for this eigenvalue λ is a subspace of n-space, the *eigenspace* of λ.

Note. The proofs of (1) and (3) are to be given in Exercises 27 and 29.

Note. We now define an eigenvalue and eigenvector for a linear transformation between general vector spaces. The definition depends only on the linearity of the transformation and so is valid even when dealing with infinite dimensional vector spaces.

Definition 5.2. Eigenvalues and Eigenvectors.

Let T be a linear transformation of a vector space V into itself. A scalar λ is an *eigenvalue* of T if there is a nonzero vector $\vec{v} \in V$ such that $T(\vec{v}) = \lambda\vec{v}$. The vector $\vec{v}$ is then an *eigenvector* of T corresponding to λ.

Note 5.1.B. In the event that T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ then there is an $n \times n$ matrix A such that $T(\vec{x}) = A\vec{x}$ for $\vec{x} \in \mathbb{R}^n$ by Corollary 2.3.A. In this case, the eigenvalues and eigenvectors of T and A coincide. The next example illustrates the use of Definition 5.2 in the setting of the infinite dimensional vector space D_∞ of all functions mapping $\mathbb{R}$ into $\mathbb{R}$ which are differentiable of all orders (see Notes 3.2.A).

Example. Page 298 Example 8. Let D_∞ be the vector space of all functions mapping $\mathbb{R}$ onto $\mathbb{R}$ and having derivatives of all order. Let $T : D_\infty \to D_\infty$ be the differentiation map so that $T(f) = f'$. Describe all eigenvalues and eigenvectors of T. (Notice that by Example 3.4.5, T actually is linear.)

Examples. Page 300 number 18, Page 301 Number 30, Page 301 Number 32, Page 302 Number 38, Page 302 Number 40.

Revised: 11/8/2018