Chapter 5. Eigenvalues and Eigenvectors

5.2 Diagonalization

Note. In this section, we define what it means to "diagonalize" a matrix and show how certain matrices can be diagonalized in terms of eigenvalues and eigenvectors. Once a matrix is diagonalized, then it can easily be raised to powers, finally addressing the applications discussed at the beginning of Section 5.1.

Recall. A matrix is diagonal if all entries off the main diagonal are 0.

Note. In this section, the theorems stated are valid for matrices and vectors with complex entries and complex scalars, unless stated otherwise.

Note. Throughout this section, the results hold for matrices with complex entries, though our examples and exercises only involve real numbers. We will have to delay some of the proofs until we deal with complex numbers in Chapter 9. One result we should state is the Fundamental Theorem of Algebra.

> **Fundamental Theorem of Algebra.** If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ is an n-degree polynomial with either real or complex coefficients, then p can be factored as $p(x) = a_n(x-r_1)(x-r_2)\cdots(x-r_n)$ where $r_1, r_2, \ldots, r_n$ are the roots of p. The roots may not be distinct and may be complex (even it the coefficients $a_0, a_1, \ldots, a_n$ are real).

For $n \times n$ matrix A, the characteristic polynomial $p(\lambda) = \det(A - \lambda\mathcal{I})$ is an n-degree polynomial. So the Fundamental Theorem of Algebra guarantees that A has n (not necessarily distinct) eigenvalues; though we need to allow complex eigenvalues for this to be the case. Recall that the *multiplicity* of a root r of a polynomial, is the number of times the factor $(x - r)$ appears in the factorization of $p(x)$ given in the Fundamental Theorem of Algebra.

Note. The following theorem is fundamental in diagonalizing a matrix. Notice that it is based on eigenvalues and eigenvectors of a matrix, giving motivation for finding eigenvalues and eigenvectors.

Theorem 5.2. Matrix Summary of Eigenvalues of A.
Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be (possibly complex) scalars and $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ be nonzero vectors in n-space. Let C be the $n \times n$ matrix having $\vec{v_j}$ as jth column vector and let

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Then $AC = CD$ if and only if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A and $\vec{v_j}$ is an eigenvector of A corresponding to λ_j for $j = 1, 2, \ldots, n$.

Note. The $n \times n$ matrix C is invertible if and only if $\operatorname{rank}(C) = n$ by Theorem 2.6, "An Invertibility Criterion"; that is, if and only if the column vectors of C form a basis of n-space. In this case, the criterion $AC = CD$ in Theorem 5.2 can be written as $D = C^{-1}AC$. The equation $D = C^{-1}AC$ transforms a matrix A into a diagonal matrix D that is much easier to raise to powers.

Definition 5.3. Diagonalizable Matrix.
An $n \times n$ matrix A is *diagonalizable* if there exists an invertible matrix C such that $C^{-1}AC = D$ is a diagonal matrix. The matrix C is said to *diagonalize* the matrix A.

Corollary 1. A Criterion for Diagonalization.
An $n \times n$ matrix A is diagonalizable if and only if n-space has a basis consisting of eigenvectors of A.

Corollary 2. Computation of A^k.
Let an $n \times n$ matrix A have n eigenvectors and eigenvalues, giving rise to the matrices C and D so that $AC = CD$, as described in Theorem 5.2. If the eigenvectors are independent, then C is an invertible matrix and $C^{-1}AC = D$. Under these conditions, we have $A^k = CD^kC^{-1}$.

Example 5.2.A. Diagonalize $A = \begin{bmatrix} 5 & -3 \\ -6 & 2 \end{bmatrix}$ and calculate A^k.

Theorem 5.3. Independence of Eigenvectors.

Let A be an $n \times n$ matrix. If $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, the set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ is linearly independent and A is diagonalizable.

Example. Page 315 Number 6.

Definition 5.4. An $n \times n$ matrix P is *similar* to an $n \times n$ matrix Q if there exists an invertible $n \times n$ matrix C such that $C^{-1}PC = Q$.

Example. Page 315 Number 18.

Definition. The *algebraic multiplicity* of an eigenvalue λ_i of A is its multiplicity as a root of the characteristic equation of A. Its *geometric multiplicity* is the dimension of the eigenspace E_{λ_i}.

Theorem 5.2.A. The geometric multiplicity of an eigenvalue of a matrix A is less than or equal to its algebraic multiplicity.

Note. The proof of Theorem 5.2.A is an exercise (Number 33) in section 9.4.

Theorem 5.4. A Criterion for Diagonalization.

An $n \times n$ matrix A is diagonalizable if and only if the algebraic multiplicity of each (possibly complex) eigenvalue is equal to its geometric multiplicity.

Example. Page 315 Number 10. In this example, we have an eigenvalue of algebraic multiplicity 3 and geometric multiplicity 1.

Theorem 5.5. Diagonalization of Real Symmetric Matrices.

Every real symmetric matrix is real diagonalizable. That is, if A is an $n \times n$ symmetric real matrix with real-number entries, then each eigenvalue of A is a real number, and its algebraic multiplicity equals its geometric multiplicity.

Note. The proof of Theorem 5.5 is in Chapter 9. See the Corollary to Theorem 9.5, "Spectral Theorem for Hermitian Matrices."

Examples. Page 316 Number 22, Page 316 Number 24, Page 316 Number 26.

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