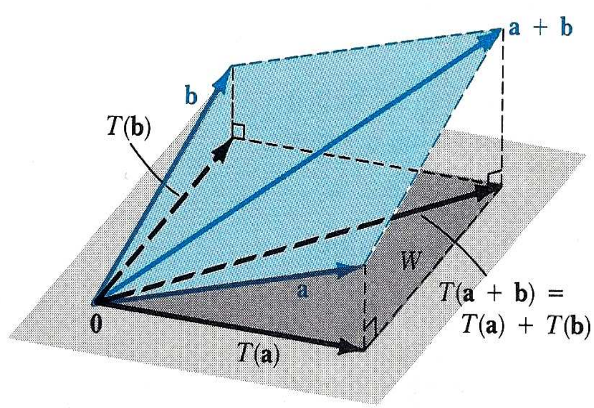
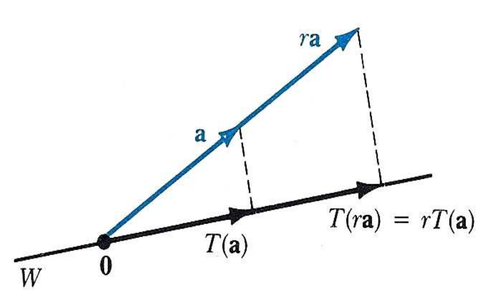
Chapter 6. Orthogonality

6.4 The Projection Matrix

Note. In Section 6.1 (Projections), we projected a vector $\vec{b} \in \mathbb{R}^n$ onto a subspace W of $\mathbb{R}^n$. We did so by finding a basis for W and a basis for the "perp space" $W^\perp$. We then found the coordinate vector of $\vec{b}$ with respect to these two bases combined, and from this the projection of $\vec{b}$ onto W could be found. In Figure 6.8, a projection of $\vec{a}+\vec{b}$ onto a subspace W is given schematically (or we can interpret this as a situation where we are in $\mathbb{R}^3$ and W is a two dimensional subspace). With the projection onto W represented as the function T, this figure suggests that $T(\vec{a}+\vec{b}) = T(\vec{a}) + T(\vec{b})$. Figure 6.9 suggests that $T(r\vec{a}) = rT(\vec{a})$.



Projection of a + b on W.



Projection of ra on W.

If these do in fact hold, then we would have that projection is a linear transformation and so (by Theorem 3.10) there would exist a matrix P such that the projection of $\vec{b}$ onto W is given by $P\vec{b}$.

Note. In this section, we show that the projection of any $\vec{b}$ onto W is of the form $P\vec{b}$ where P is a matrix independent of $\vec{b}$. This will then show that projection onto W is a linear transformation. We first need a preliminary theorem on rank.

Theorem 6.10. The Rank of $(A^T)A$.
Let A be an $m \times n$ matrix of rank r. Then the $n \times n$ symmetric matrix $(A^T)A$ also has rank r.

Note. Let $W = \text{sp}(\vec{a}_1, \vec{a}_2, \ldots, \vec{z}_k)$ be a subspace of $\mathbb{R}^n$ where $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k$ are linearly independent (so $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k\}$ is a basis for W, though maybe neither an orthogonal nor orthonormal basis for W). Let $\vec{b} \in \mathbb{R}^n$ and let $\vec{p} = \vec{b}_W$ be the projection of $\vec{b}$ onto W. Then $\vec{b} = \vec{p} + (\vec{b} - \vec{p}) - \vec{b}_W + \vec{b}_{W^\perp}$. By Theorem 6.1(4), $\vec{p}$ is the unique vector such that:

1. the vector $\vec{p}$ lies in the subspace W, and

2. the vector $\vec{b} - \vec{p}$ is perpendicular to every vector in $W^\perp$ (or $\vec{b} - \vec{p} \in W^\perp$).

Note. Let A be an $n \times k$ matrix with its columns as $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k$. Then the column space of A is W. Since $\vec{p} \in W$ then (by the "Column Space Criterion" of Section 1.6, see page 92) there is column vector $\vec{r} \in \mathbb{R}^k$ such that $\vec{p} = A\vec{r}$. Since $\vec{b} - \vec{p} = \vec{b} - A\vec{r}$ is perpendicular to each vector in W, then $(A\vec{x}) \cdot (\vec{b} - A\vec{r}) = 0$ for any $A\vec{x} \in W$ (where $\vec{x} \in \mathbb{R}^k$ is any column vector). So

$$(A\vec{x}) \cdot (\vec{b} - A\vec{r}) = (A\vec{x})^T(\vec{b} - A\vec{r}) = \vec{x}^T A^T(\vec{b} - A\vec{r}) = \vec{x}^T(A^T\vec{b} - A^T A\vec{r}) = [0].$$

But this means that $\vec{x} \cdot (A^T\vec{b} - A^T A\vec{r}) = 0$ for *any* $\vec{x} \in \mathbb{R}^k$. Since $A^T\vec{b} - A^T A\vec{r} \in$

$\mathbb{R}^k$, this implies that $A^T\vec{b} - A^TA\vec{r} = \vec{0} \in \mathbb{R}^k$ (this follows from Exercise 41(c) of Section 1.3). Now $k \times k$ matrix A^TA is invertible since A is invertible (by Theorem 1.12, since its columns are linearly independent), so rank$(A) = k$ by Theorem 2.6 ("An Invertibility Criterion') and by Theorem 6.10 ("The Rank of $(A^T)A$"), rank$(A^TA) = k$ and so, again by Theorem 2.6, A^TA is invertible. So we have $A^T\vec{b} - A^TA\vec{r} = \vec{0}$ implies that $\vec{r} = (A^TA)^{-1}(A^T\vec{b})$. Since $\vec{p} = A\vec{r} = \vec{b}_W$ then $\vec{b}_W = A(A^TA)^{-1}A^T\vec{b}$.

Note. We summarize the argument above as:

<u>Projection $\vec{b}_W$ of $\vec{b}$ onto the Subspace W</u>

Let $W = \text{sp}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k)$ be a k-dimensional subspace of $\mathbb{R}^n$ and let matrix A have as its columns the vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k$. The projection of $\vec{b}$ onto W is $\vec{b}_W = A(A^TA)^{-1}A^T\vec{b}$.

Definition. Let $W = \text{sp}(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k)$ be a k-dimensional subspace of $\mathbb{R}^n$ and let matrix A have as its columns the vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k$. The matrix $P = A(A^TA)^{-1}A^T$ is the *projection matrix* for the subspace W.

Example. Page 363 Example 6.4.2.

Example. Page 368 number 6.

Note. We see that the projection matrix P is computed in terms of matrix A which is based on the basis for W. However, the choice of P is ultimately unique, as the next theorem claims.

Theorem 6.11. Projection Matrix.
Let W be a subspace of $\mathbb{R}^n$. There is a unique $n \times n$ matrix P such that, for each column vector $\vec{b} \in \mathbb{R}^n$, the vector $P\vec{b}$ is the projection of $\vec{b}$ onto W. The projection matrix can be found by selecting any basis $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k\}$ for W and computing $P = A(A^T A)^{-1} A^T$, where A is the $n \times k$ matrix having column vectors $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k$.

Theorem 6.12. Characterization Projection Matrices.
The projection matrix P for a subspace W of $\mathbb{R}^n$ is both idempotent (that is, $P^2 = P$) and symmetric (that is, $P = P^T$). Conversely, every $n \times n$ matrix that is both idempotent and symmetric is a projection matrix (specifically, it is the projection matrix for its column space).

Note. Since Theorem 6.12 says that (to paraphrase) "P is a projection matrix if and only if P is idempotent and symmetric," then we could take this as the definition of a projection matrix. There may be setting out there where you see this as the definition. Unfortunately, such a definition looses site of all the underlying geometric properties which we have used here.

Note. If we have an orthonormal basis for W, say $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k\}$, then Fraliegh and Beauregard claim that $A^T A = \mathcal{I}$ where $\mathcal{I}$ is the $k \times k$ identity. This follows from the fact that the (ij)-entry of $A^T A$ is the ith row vector of A^T dotted with the jth column vector of A. But this is simply $\vec{a}_i \cdot \vec{a}_j$. Since the basis is orthonormal, $\vec{a}_i \cdot \vec{a}_j = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j. \end{cases}$ So $A^T A = \mathcal{I}$. Hence

$$P = A(A^T A)^{-1} A^T - A\mathcal{I}A^T = AA^T.$$

So the computation of P is simplified when we have an orthonormal basis for W.

Examples. Page 369 numbers 28 and 32.

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