Chapter 8. Eigenvalues: Further Applications and Computations

8.1. Diagonalization of Quadratic Forms

Note. In this section we define a quadratic form and relate it to a vector and matrix product. We define diagonalization of a quadratic form and give an algorithm to diagonalize a quadratic form. The fact that every quadratic form can be diagonalized (using an orthogonal matrix) is claimed by the "Principal Axis Theorem" (Theorem 8.1). It is applied in Section 8.2 to address quadratic surfaces.

Definition. A *quadratic form* $f(\vec{x}) = f(x_1, x_2, \ldots, x_n)$ in n variables is a polynomial that can be written as

$$f(\vec{x}) = \sum_{i,j=1;\, i\leq j}^{n} u_{ij}x_ix_j$$

where not all u_{ij} are zero.

Note. For $n = 2$, a quadratic form is $f(x, y) = ax^2 + bxy + xy^2$. The graph of such an equation in the xy-plane is a conic section (ellipse, hyperbola, or parabola). For $n = 3$, we have

$$f(x_1, x_2, x_3) = u_{11}x_1^2 + u_{12}x_1x_2 + u_{13}x_1x_3 + u_{22}x_2^2 + u_{23}x_2x_3 + u_{33}x_2^2.$$

We'll see in the next section that the graph of $f(x_1, x_2, x_3)$ is a quadric surface. We can easily verify that

$$[f(x_1, x_2, x_3)] = [x_1, x_2, x_3] \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Definition. In the quadratic form $\vec{x}^T U \vec{x}$, the matrix U is the *upper-triangular coefficient matrix* of the quadratic form.

Example. Page 417 Number 4(a).

Note. In fact, every quadratic form can be written as $\vec{x}^T A \vec{x}$ where A is symmetric. We take $a_{ii} = u_{ii}$ for $i = 1, 2, \ldots, n$ and $a_{ij} = a_{ji} = u_{ij}/2$ for $i < j$.

Definition. If $\vec{x}^T A \vec{x}$ is a quadratic form where A is a symmetric matrix then A is the *symmetric coefficient matrix* of the quadratic form.

Example. Page 417 Number 4(b).

Note. By Theorem 6.8, "Fundamental Theorem of Real Symmetric Matrices" of Section 6.3, for $n \times n$ symmetric matrix A, there is an $n \times n$ orthogonal matrix C such that $C^{-1}AC = D$ where D is a diagonal matrix. By Theorem 5.2, "Matrix Summary of Eigenvalues of A," in Section 5.2, the diagonal entries of D are the eigenvalues of A and the jth column of C is a unit eigenvector (since C is orthogonal) of A corresponding to eigenvalue $\lambda_i = d_{ii}$. So if $\vec{x}^T A \vec{x}$ is a quadratic form where A is symmetric then the substitution $\vec{x} = C\vec{t}$ (or "change of variables") yields

$$\begin{aligned}
\vec{x}^T A \vec{x} &= (C\vec{t})^T A (C\vec{t}) = \vec{t}^T C^T A C \vec{t} \\
&= \vec{t}^T C^{-1} A C \vec{t} \text{ since } C \text{ is orthogonal then } C^T = C^{-1} \\
&\quad\ \text{(see Definition 6.4)} \\
&= \vec{t}^T D \vec{t} = \lambda - 1 t_1^2 + \lambda_2 t_2^2 + \cdots + \lambda_n t_n^2.
\end{aligned}$$

Definition. Quadratic form $\vec{x}^T A \vec{x}$ is *diagonalized* when the substitution $\vec{x} = C\vec{t}$ leads to the equality $\vec{x}^T A \vec{x} = \lambda_1 t_1^2 + \lambda_2 t_2^2 + \cdots + \lambda_n t_n^2$.

Note. With quadratic form $\vec{x}^T A \vec{x}$, where A is symmetric, diagonalized by $\vec{x} = C\vec{t}$, then for all $\vec{x} \in \mathbb{R}^n$ we have $\vec{t} = C^{-1}\vec{x}$ that

$$\vec{t}^T D \vec{t} = (C^{-1}\vec{x})^T D (C^{-1}\vec{x}) = \vec{x}^T (C^{-1})^T D C^{-1} \vec{x} = \vec{x}^T C D C^{-1} \vec{x} = \vec{x}^T A \vec{x}.$$

Example. Page 417 Number 12.

Note. We summarize the above computations in the following theorem.

Theorem 8.1. Principal Axis Theorem.
Every quadratic form $f(\vec{x})$ in n variables $x_1, x_2, \ldots, x_n$ can be diagonalized by a substitution $\vec{x} = C\vec{t}$ where C is an $n \times n$ orthogonal matrix. The diagonalized form appears as $\lambda_1 t_1^2 + \lambda_2 t_2^2 + \cdots \lambda_n t_n^2$, where the λ_j are the eigenvalues of the symmetric coefficient matrix A of $f(\vec{x})$. The jth column of C is a normalized eigenvector $\vec{v}_j$ of A corresponding to λ_j. Moreover, C can be chosen so that $\det(C) = 1$.

Note. By Page 359 Number 22, since C is orthogonal we have $\det(C) = \pm 1$. We can think of C as a change-of-coordinates matrix from the basis of $\mathbb{R}^n$ of eigenvectors to the standard basis. We claim "without proof" (Fraleigh and Beauregard, page 413) that $\det(C)$ is related to the "orientation" of the ordered basis of eigenvectors versus the orientation A the standard basis vectors. Two ordered bases have the same orientation if one set of basis vectors (in order) can be rotated onto the other set of basis vectors (in order). Otherwise the ordered bases have opposite orientation. Figure 8.1(a) below shows two ordered bases in $\mathbb{R}^2$ that have the same orientation. Figure 8.1(b) shows two ordered bases in $\mathbb{R}^2$ that have opposite orientations. When $\det(C) = 1$ the basis of eigenvectors and the standard basis have the same orientation; if $\det(C) = -1$ then they have opposite orientations.

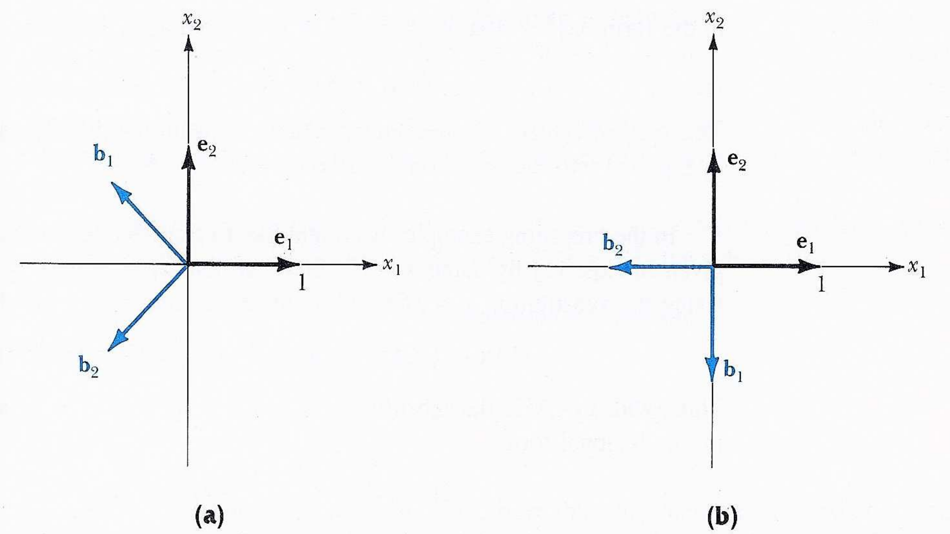

FIGURE 8.1 **(a) Rotation of axes, hence ($\mathbf{b}_1$, $\mathbf{b}_2$) and ($\mathbf{e}_1$, $\mathbf{e}_2$) have the same orientation; (b) not a rotation of axes, hence, ($\mathbf{b}_1$, $\mathbf{b}_2$) and ($\mathbf{e}_1$, $\mathbf{e}_2$) have opposite orientation.**