Chapter 8. Eigenvalues: Further Applications and Computations

8.2. Applications to Geometry

Note. Figure 8.2 shows there intersections of planes with a double right-circular cone ("right-circular" because a plane perpendicular to the axis of the cone cuts the cone in a circle). The resulting "conic sections" shown are an ellipse (a circle is a special case of an ellipse), a hyperbola, and a parabola. In this section we state the standard equations for these three conic sections without any derivation. For an alternated approach where definitions are given in terms of sums and differences of certain distances (and the derivation of our formulae follow from these definitions) see my online Calculus 3 notes: `http://faculty.etsu.edu/gardnerr/2110/notes-12e/c11s6.pdf`.

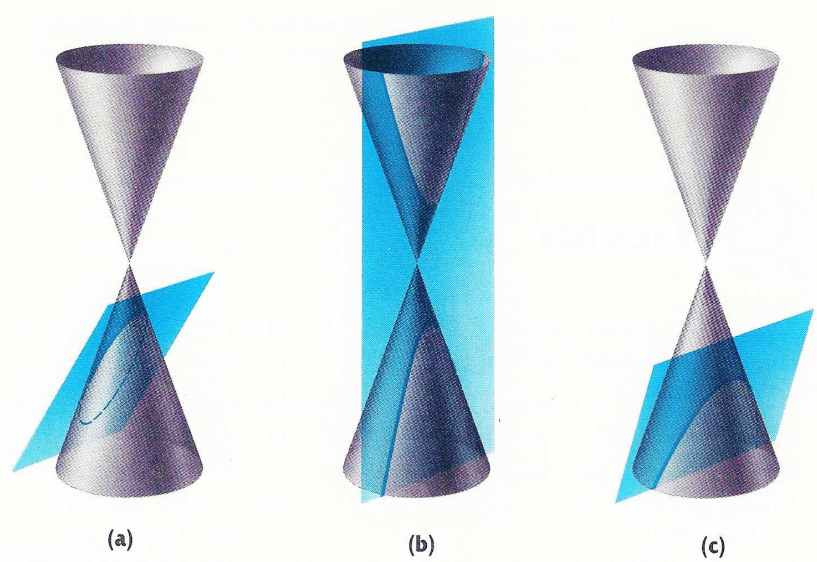

FIGURE 8.2 Sections of a cone: (a) an elliptic section; (b) a hyperbolic section; (c) a parabolic section.

Note. The equation of an ellipse in standard form (that is, with center at $(0,0)$) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

If the center of the ellipse is at the point (h,k) then the equation is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

See Figure 8.3.

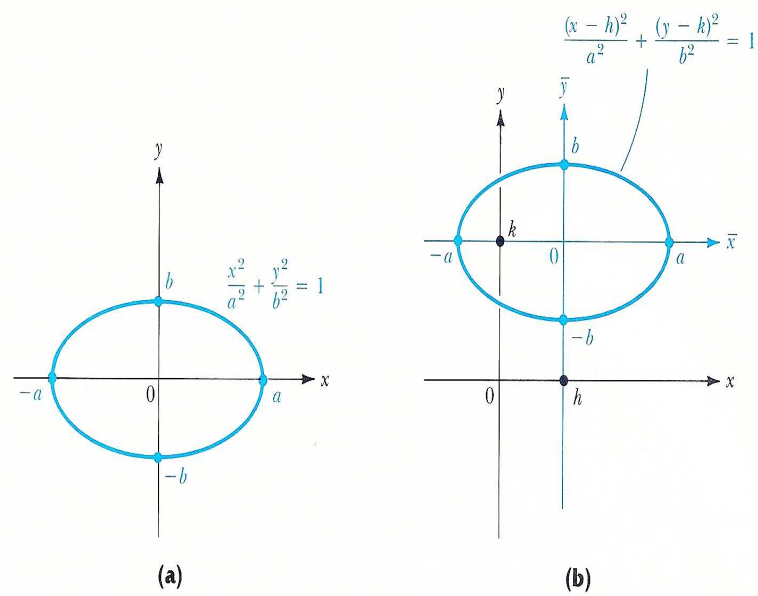

FIGURE 8.3 (a) Ellipse in standard position; (b) ellipse centered at (h, k).

Note. Every polynomial equation of the form $c_1x^2 + c_2y^2 + c_3x + c_4y = d$, where nonzero c_1 an c_2 are of the same sign, determines an ellipse. The center can be found by completing the square of the x^2 and x terms and of the y^2 and y terms. It's possible for the equation to become of the form $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 0$ in which case the solution set is the single point (h,k); this is called a *degenerate ellipse*. If the equation can be put in the form $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = e < 0$ then this is an *empty ellipse*.

Note. Every polynomial equation of the form $c_1x^2 + c_2y^2 + c_3x + c_4y = d$, where nonzero c_1 an c_2 are of opposite signs, determines a hyperbola. The standard form of the equation of a hyperbola (with center $(0,0)$) is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ or } -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

These hyperbolas have asymptotes of $y = \pm(b/a)x$; see Figure 8.4. If the center is not $(0,0)$ then completing the square (as with the ellipse) allows us to equations of the forms

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \text{ or } -\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

As with ellipses, a hyperbola may be degenerate (in which case the hyperbola consists of the straight lines $y = \pm(b/a)x$). There is no empty hyperbola.

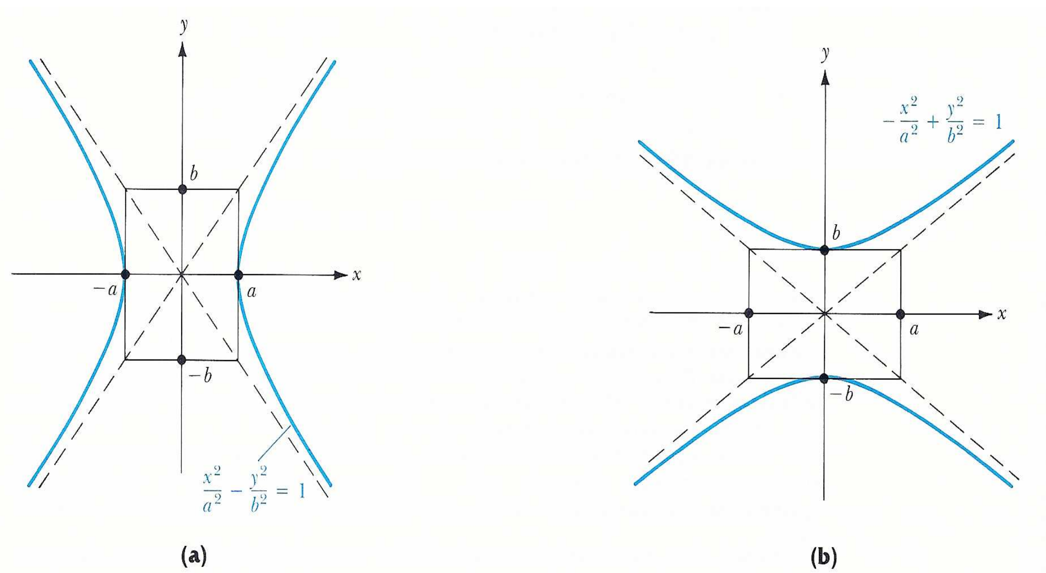

FIGURE 8.4 (a) The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; (b) the hyperbola $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Note. A polynomial equation of the form

$$c_1x^2 + c_2x + c_3y = d \text{ or } c_1y^2 + c_2x + c_3y = d \text{ where } c_1 \neq 0,$$

determine a parabola. The standard form of the equation of a parabola (with vertex at $(0,0)$) is $ay = x^2$ or $ax = y^2$ (see Figure 8.5). A parabola can can be degenerate (consisting only of the vertex, say when parameter $a = 0$ here) or empty (where parameter $a < 0$ here). Of course a parabola may have its vertex at a point (h,k) other than the origin.

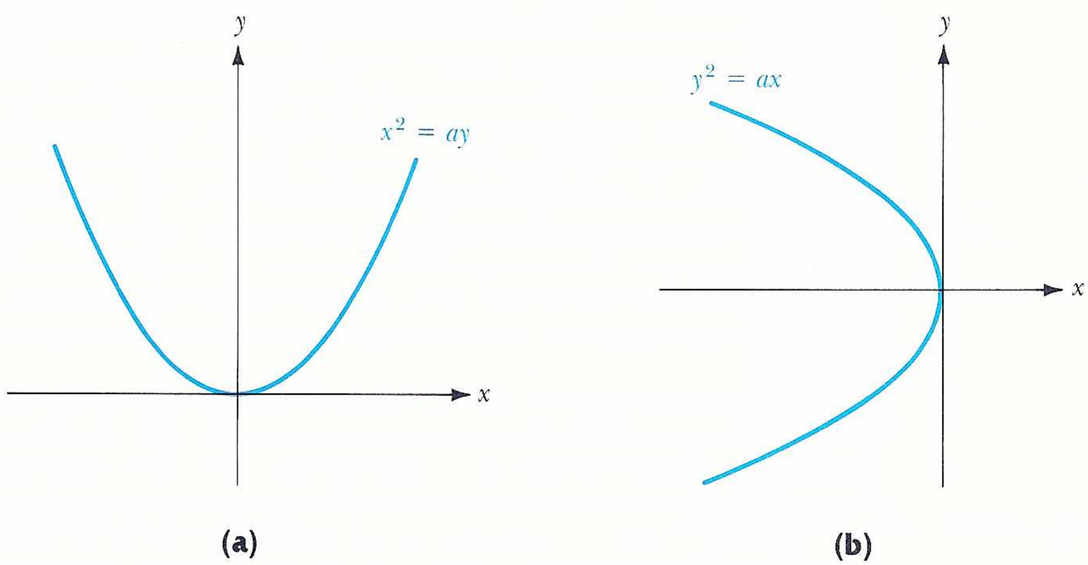

FIGURE 8.5 (a) The parabola $x^2 = ay$, $a > 0$; (b) the parabola $y^2 = ax$, $a < 0$.

Note. So, every equation of the form $c_1x^2 + c_2y^2 + c_3x + c_4y = d$ with at least one of c_1 or c_2 nonzero describes a (possibly degenerate or empty) ellipse, hyperbola, or parabola.

Note. We now turn our attention to equations of the form

$$ax^2 + bxy + xy^2 + dx + ey + f = 0 \text{ for } a, b, c \text{ not all zero.}$$

We use the results of the previous section to "transform away" the xy term, while still preserving the shape of the curve determined by the equation.

Theorem 8.2. Classification of Second-Degree Plane Curves.
Every equation of the form

$$ax^2 + bxy + xy^2 + dx + ey + f = 0 \text{ for } a, b, c \text{ not all zero}$$

can be reduced to an equation of the form

$$\lambda_1 t_1^2 + \lambda_2 t_2^2 + gt_1 + ht_2 + k = 0$$

by means of an orthogonal substitution corresponding to a rotation of the plane. The coefficients λ_1 and λ_2 in the second equation are the eigenvalues of the symmetric coefficient matrix of the quadratic-form portion of the first equation. The curve describes a (possibly degenerate or empty)

$$\begin{array}{ll} \text{ellipse} & \text{if } \lambda_1\lambda_2 > 0 \\ \text{hyperbola} & \text{if } \lambda_1\lambda_2 < 0 \\ \text{parabola} & \text{if } \lambda_1\lambda_2 = 0. \end{array}$$

Page 422 Example 8.2.2. Use rotation and translation of axes to sketch the curve $2xy + 2\sqrt{2}x = 1$.

Solution. To find the symmetric coefficient matrix for the cross term

$$2xy = 0x^2 + 1xy + 1yx + 0y^2 = \frac{u_{11}}{2}x^2 + \frac{u_{12}}{2}xy + \frac{u_{21}}{2}yx + \frac{u_{22}}{2}y^2,$$

we take $u_{11} = u_{22} = 0$ and $u_{12} = u_{21} = 2$. Since $a_{ij} = a_{ji} = u_{ij}/2$ by Theorem 8.1.A, we have $a_{11} - a_{22} = 0$ and $a_{12} = a_{21} = 1$ so that the symmetric coefficient matrix is $A = \begin{bmatrix} 1_{11} & 1_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. For the eigenvalues of A consider

$$\det(A - \lambda\mathcal{I}) = \begin{vmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

and so the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 1$. Since $\lambda_1\lambda_2 = -1 < 0$ then by Theorem 8.2, the equation determines a hyperbola. We need a basis for $\mathbb{R}^2$ consisting of normalized eigenvectors of A (Step 3 in "Diagonalizing a Quadratic Form $f(\vec{x})$" of Section 8.1). For $\lambda_1 = -1$ and $\vec{v}_1 = [v_1, v_2]^T$ a corresponding eigenvector, we need $(A - \lambda_1\mathcal{I})\vec{v}_1 = \vec{0}$, so consider

$$[a - \lambda_1\mathcal{I} \mid \vec{0}] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \underset{\sim}{\overset{R_2 \to R_2 - R_1}{}} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and so we take $v_1 = -v_2$ and to normalize $\vec{v}_1$ we choose $\vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$. Similiarly, for $\lambda_2 = 1$ and $\vec{v}_2 = [v_1, v_2]^T$ a correpsonding eigenvector, we consider

$$[A - \lambda_1\mathcal{I} \mid \vec{0}] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \underset{\sim}{\overset{R_2 \to R_2 + R_1}{}} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \underset{\sim}{\overset{R_1 \to -R_1}{}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and so we take $v_1 = v_2$ and to normalize $\vec{v}_2$ we choose $\vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. So $C = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ (notice that $\det(C) = 1$). As seen in the proof of the

"Classification of Second-Degree Plane Curves," Theorem 8.2, we take

$$\begin{bmatrix} x \\ y \end{bmatrix} = C \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} (1/\sqrt{2})(t_1 + t_2) \\ (1/\sqrt{2})(-t_1 - t_2) \end{bmatrix}.$$

So with $x = (1/\sqrt{2})(t_1 + t_2)$ and $y = (1/\sqrt{2})(-t_1 + t_2)$ we have that $2xy + 2\sqrt{2}x = 1$ becomes $2\frac{1}{\sqrt{2}}(t_1+t_2)\frac{1}{\sqrt{2}}(-t_1+t_2) + 2\sqrt{2}\frac{1}{\sqrt{2}}(t_1+t_2) = 1$ or $(t_2^2 - t_1^2) + 2(t_1 + t_2) = 1$ or $-t_1^2 + 2t_1 + {}_2^2 + 2t_2 = 1$ or $(t_2+1)^2 - (t_1-1)^2 = 1$. So this is in fact a hyperbola in the (t_1, t_2)-coordinate system. Notice that $C\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ and $C\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ so C rotates the x-axis $-45°$ so that the t_1-axis lies on the line $y = -x$ and C rotates the y-axis $-45°$ so that the t_2-axis lies on the line $y = x$. The center of hyperbola $(t_2+1)^2 - (t_1-1)^2 = 1$ is then $(1, -1)$ in the (t_1, t_2)-coordinate system. The graph is as given in Figure 8.6. However, in the text, the eigenvalues are labeled $\lambda_1 = 1$ and $\lambda_2 = -1$ so that the t_1- and t_2-axes of Figure 8.6 do not correspond to our t_1- and t_2-axis (rotate the t_1- and t_2-axes of Figure 8.6 by $-90°$ to get our axes).

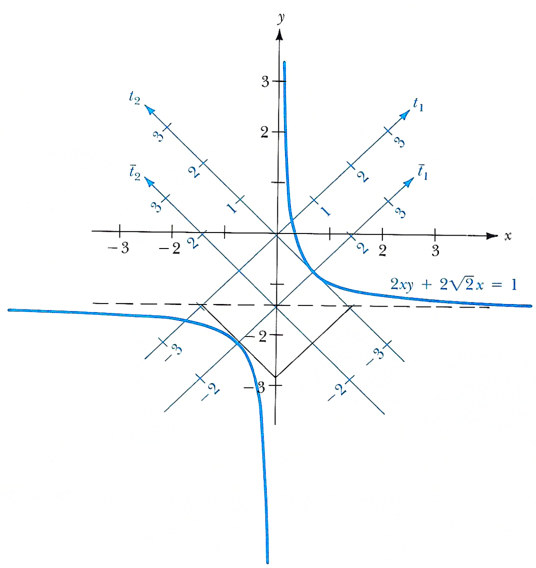

FIGURE 8.6 **The hyperbola $2xy + 2\sqrt{2}x = 1$.**

Definition. An equation in three variables of the form

$$c_1x^2 + c_2y^2 + c_3z^2 + c_4z + c_5y + c_6z = d,$$

where at least one of c_1, c_2, or c_3 in nonzero, describes a *quadric surface* in 3-space (which might be degenerate or empty).

Note. Figures 8.7 through 8.15 (see the next page) show the eight possible quadric surfaces, along with equations. The equations are for quadric surfaces centered at the origin $(0, 0, 0)$. By completing squares, we can convert these into quadric surfaces centered at point (h, k, m).

Note. Every equation of the form

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + pz + qy + rz + s = 0$$

can be transformed into one of the forms of a quadric surface given in Figures 8.7 through 8.15. The technique is similar to that given in the proof of Theorem 8.2, "Classification of Second-Degree Plane Curves." We summarize the claim in the following theorem.

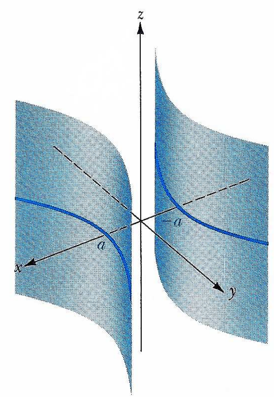

FIGURE 8.8 The hyperbolic cylinder $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

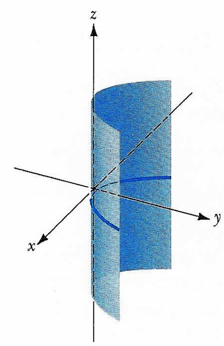

FIGURE 8.9 The parabolic cylinder $ay = x^2$.

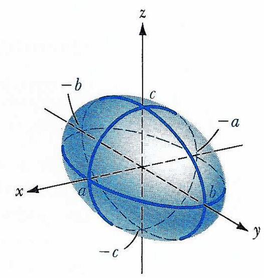

FIGURE 8.10 The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

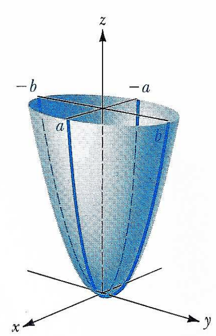

FIGURE 8.11 The elliptic paraboloid $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.

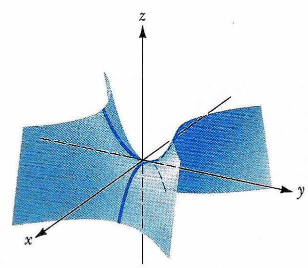

FIGURE 8.12 The hyperbolic paraboloid $z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$.

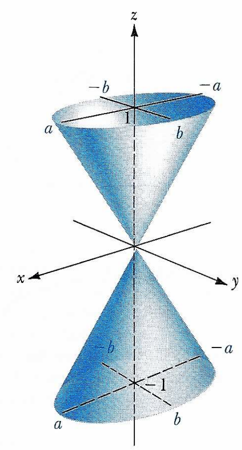

FIGURE 8.13 The elliptic cone $z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.

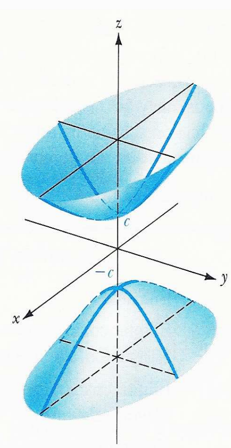

FIGURE 8.14 The hyperboloid of two sheets $\frac{z^2}{c^2} - 1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.

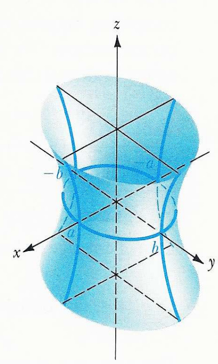

FIGURE 8.15 The hyperboloid of one sheet $\frac{z^2}{c^2} + 1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.

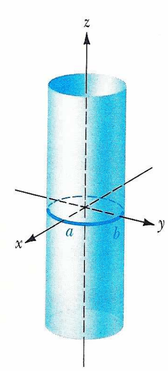

FIGURE 8.7 The elliptic cylinder $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Theorem 8.3. Principal Axis Theorem for $\mathbb{R}^3$.

Every equation of the form

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + pz + qy + rz + s = 0$$

can be reduced to an equation of the form

$$\lambda_1 t_1^2 + \lambda_2 t_2^2 + \lambda_3 t_3^2 + p' t_1 + q' t_2 + r' t_3 + s' = 0$$

by an orthogonal substitution that corresponds to a rotation of axes.

Note. As in Theorem 8.2, we can classify which quadric surface is determined by

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + pz + qy + rz + s = 0$$

in terms of the eigenvalues λ_1, λ_2, and λ_3, we have the following (though some ambiguity remains):

Eigenvalues $\lambda_1, \lambda_2, \lambda_3$	**Quadric Surface**
All of the same sign	Ellipsoid
Two of one sign and one of the other sign	Elliptic cone, hyperboloid of two sheets, or hyperboloid of one sheet
One zero, two of the same sign	Elliptic paraboloid or elliptic cylinder (degenerate case)
One zero, two of opposite signs	Hyperbolic paraboloid or hyperbolic cylinder (degenerate case)
Two zero, one nonzero	Parabolic cylinder or two parallel planes (degenerate case)

Example. Page 430 Number 16.