Chapter 9. Complex Scalars

9.2. Matrices and Vector Spaces with Complex Scalars

Note. In this section we consider the material from earlier chapters but instead of using real numbers for the entries of matrices and vectors, we use complex numbers. Most of the vector space properties and techniques carry over unmodified. We will need a new definition for inner product and we introduce a new operation on a complex matrix related to the operation of taking a transpose. The real benefit of using the complex numbers will be seen in the next two sections when we consider eigenvalues (again) and the Jordan canonical form of a matrix.

Note. We define the $n$ dimensional complex vector space $\mathbb{C}^n$ as the set of all $n$-tuples of complex numbers,

$$\mathbb{C}^n = \{[z_1, z_2, \ldots, z_m] \mid z_k \in \mathbb{C} \text{ for } k = 1, 2, \ldots, n\}.$$

Vector addition and scalar multiplication are similar to the $\mathbb{R}^n$ setting. The standard basis is

$$\{\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n\} = \{[1, 0, 0, \ldots, 0, 0], [0, 1, 0, \ldots, 0],$$

$$\ldots, [0, 0, 0, \ldots, 0, 1, 0], [0, 0, 0, \ldots, 0, 0, 1]\}.$$

We start with two examples to show that the manipulations are the same as before.

Examples. Page 472 Number 8 and Page 473 Number 10.
Note. Additional illustrations of the parallel behavior between the real and complex settings are given in Example 1 (page 465; solving a system of equations), Example 3 (page 466; showing a set of complex vectors is a basis for $\mathbb{C}^3$), and Example 4 (page 466; finding a coordinate vector relative to an ordered basis).

Note. We now give a definition of inner product on $\mathbb{C}^n$. Although the definition below looks different from the definition of the inner product on $\mathbb{R}^n$ (or the “dot product”; see Definition 1.6), when applied to vectors with real components it produces the same result as the real inner product.

**Definition 9.1.** Let $\vec{u} = [u_1, u_2, \ldots, u_n]$ and $\vec{v} = [v_1, v_2, \ldots, v_n]$ be vectors in $\mathbb{C}^n$. The *Euclidean inner product* of $\vec{u}$ and $\vec{v}$ is

$$
\langle \vec{u}, \vec{v} \rangle = \overline{u}_1 v_1 + \overline{u}_2 v_2 + \ldots + \overline{u}_n v_n = \sum_{k=1}^{n} \overline{u}_k v_k
$$

where the overline represents complex conjugation.

**Note.** For $\vec{u} \in \mathbb{C}^n$, $\langle \vec{u}, \vec{u} \rangle = |u_1|^2 + |u_2|^2 + \cdots + |u_n|^2$ where $|u_k|$ is the modulus of complex number $u_k$. Notice that $\langle \vec{u}, \vec{u} \rangle \geq 0$. Some other properties are given in the next theorem.

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{C}^n$ and let $z$ be a complex scalar. The:

1. $\langle \vec{u}, \vec{u} \rangle \geq 0$ and $\langle \vec{u}, \vec{u} \rangle = 0$ if and only if $\vec{u} = 0$,

2. $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$,

3. $\langle (\vec{u} + \vec{v}), \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$,

4. $\langle \vec{w}, (\vec{u} + \vec{v}) \rangle = \langle \vec{w}, \vec{u} \rangle + \langle \vec{w}, \vec{v} \rangle$,

5. $\langle z\vec{u}, \vec{v} \rangle = \overline{z} \langle \vec{u}, \vec{v} \rangle$ and $\langle \vec{u}, z\vec{v} \rangle = z \langle \vec{u}, \vec{v} \rangle$.

Note. The proofs of Theorem 9.2 parts (2) and (5) are to be given in Exercises 9.2.17 and 9.2.19.

Definition. For $\vec{v} = [v_1, v_2, \ldots, v_n] \in \mathbb{C}^n$, the norm is $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$. Vectors of norm 1 are unit vectors. Vectors $\vec{u}, \vec{v} \in \mathbb{C}^n$ are perpendicular (or orthogonal) if $\langle \vec{u}, \vec{v} \rangle = 0$; $\vec{u}$ and $\vec{v}$ are parallel if $\vec{u} = z\vec{v}$ for some scalar $z \in \mathbb{C}$.

Note. The modification of inner product from the real setting to the complex setting (and the resulting noncommutivity as given in Theorem 9.2(2)) requires that we deal with the Gram-Schmidt process in a more careful way. Let \( \vec{u}, \vec{v} \in \mathbb{C}^n \).

Notice that \( \vec{v} \) and \( \vec{w} = \vec{u} - \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v} \) are orthogonal because

\[
\langle \vec{w}, \vec{v} \rangle = \langle \vec{u} - \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle - \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{v}, \vec{v} \rangle} \langle \vec{v}, \vec{v} \rangle, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle - \langle \vec{v}, \vec{u} \rangle \langle \vec{v}, \vec{v} \rangle, \vec{v} \rangle \text{ by Theorem 9.2(3)}
\]

\[
= \langle \vec{u}, \vec{v} \rangle - \langle \vec{v}, \vec{u} \rangle \langle \vec{v}, \vec{v} \rangle, \vec{v} \rangle \text{ by Theorem 9.2(5)}
\]

\[
= \langle \vec{u}, \vec{v} \rangle - \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{v}, \vec{v} \rangle} \langle \vec{v}, \vec{v} \rangle, \vec{v} \rangle \text{ by Theorem 9.2(4), since } \langle \vec{v}, \vec{v} \rangle \text{ is real}
\]

\[
= \langle \vec{u}, \vec{v} \rangle - \langle \vec{v}, \vec{u} \rangle \text{ by Theorem 9.2(2)}
\]

\[
= 0,
\]

but it may not be the case that \( \vec{v} \) is perpendicular to \( \vec{w} = \vec{u} - \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v} \). So we interpret

\[
\frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v}
\]

as the “projection of vector \( \vec{u} \) onto vector \( \vec{v} \).” This requires a revision of the Gram-Schmidt Process of Section 6.2. We start with basis \( \{ \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k \} \) of subspace \( W \) of \( \mathbb{C}^n \) (so \( k \leq n \)) and produce orthogonal basis \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \} \) recursively as

\[
\vec{v}_j = \vec{a}_j - \left( \frac{\langle \vec{v}_1, \vec{a}_j \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 + \frac{\langle \vec{v}_2, \vec{a}_j \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \vec{v}_2 + \cdots + \frac{\langle \vec{v}_{j-1}, \vec{a}_j \rangle}{\langle \vec{v}_{j-1}, \vec{v}_{j-1} \rangle} \vec{v}_{j-1} \right).
\]

Example. Page 473 Number 28.
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Note. We now consider manipulations of matrices based on conjugation of the entries.

Definition 9.2. Conjugate Transpose and Hermitian Adjoint.
Let \( A = [a_{ij}] \) be an \( m \times n \) matrix with complex entries.

1. The \textit{conjugate} of \( A \) is the \( m \times n \) matrix \( \overline{A} = [\overline{a}_{ij}] \).

2. The \textit{conjugate transpose} (or \textit{Hermitian adjoint}) of \( A \) is the \( n \times n \) matrix \( A^* = [\overline{a}_{ij}]^T \).

Note. Recall that for \( \vec{x}, \vec{y} \in \mathbb{R}^n \) that the dot product is \( \vec{x} \cdot \vec{y} = \vec{x}^T \vec{y} \) (as a matrix product). Similarly, for \( \vec{v}, \vec{w} \in \mathbb{C}^n, \langle \vec{v}, \vec{w} \rangle = \vec{v}^* \vec{w} \).

Example. Page 473 Number 30(a). If \( A = \begin{bmatrix} 1 & 1 + i & 2 \\ 1 - i & 1 + i & 2 \end{bmatrix} \) then

\[
A^* = \begin{bmatrix} 1 & 1 - i \\ 1 - i & 1 + i \\ \overline{1 + i} & \overline{1 + i} \\ \overline{1 - i} & \overline{1 + i} \\
\end{bmatrix} = \begin{bmatrix} 1 & 1 + i \\ 1 - i & 1 - i \\ 2 & 2 \end{bmatrix}.
\]

Let $A$ and $B$ be $m \times n$ matrices. Then

1. $(A^*)^* = A$,
2. $(A + B)^* = A^* + B^*$,
3. $(zA)^* = \overline{z}A^*$ for an scalar $z \in \mathbb{C}$,
4. If $A$ and $B$ are square matrices, then $(AB)^* = B^*A^*$.

Note. If the columns of complex matrix $A$ form an orthonormal set then the $(i, j)$ entry of $A^*A$ is $\langle \vec{v}, \vec{w} \rangle$ where $\vec{v}$ is the $i$th row of $A^*$ and $\vec{w}$ is the $j$th column of $A$. But the $i$th row of $A^*$ is the conjugate of the $i$th column of $A$ and so (see Note 9.2.A):

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^* \vec{w} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}.$$ 

That is, $A^*A = I$. This inspires the following definition (which is the terminology in the complex setting which is analogous to the term “orthogonal matrix” in the real setting; see Definition 6.4, “Orthogonal Matrix,” and notice that if $A$ has real entries the $A^* = A^T$).

Definition 9.3. Unitary Matrix.

A square matrix $U$ with complex entries is unitary if its column vectors are orthogonal unit vectors; that is, if $U^*U = I$. 


Note. Recall that real matrix $A$ is symmetric if $A = A^T$ (see Definition 1.11, “Transpose of a Matrix; Symmetric Matrix”). We now generalize the idea of symmetry to the complex setting. The motivation for this definition will be seen in the next section.

Definition 9.4. Hermitian Matrix.

A square matrix $H$ is Hermitian if $H^* = H$; that is, if $H$ is equal to its conjugate transpose.

Note. If $H$ is Hermitian then all diagonal entries must be real.

Examples. Page 474 Number 34, Page 474 Number 38.

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