Chapter 9. Complex Scalars

9.3. Eigenvalues and Diagonalization

Note. In this section we consider eigenvalues and eigenvectors for matrices with complex entries. We extend the idea of diagonalization from Chapter 6 to the complex setting and prove that every Hermitian matrix is diagonalizable by a unitary matrix (in the "Spectral Theorem for Hermitian Matrices," Theorem 9.5).

Note. Recall form Section 6.3, "Orthogonal Matrices," that every real symmetric matrix is diagonalizable using an orthogonal matrix:

> **Theorem 6.8. Fundamental Theorem of Real Symmetric Matrices.**
> Every real symmetric matrix A is diagonalizable. The diagonalization $C^{-1}AC = D$ can be achieved by using a real orthogonal matrix C.

Note. The following definitions are the same as for the real setting. The computations are the same as well.

Definition. Let A be an $n \times n$ complex matrix. If $A\vec{v} = \lambda\vec{v}$ where $\lambda \in \mathbb{C}$ and $\vec{v} \in \mathbb{C}^n$, $\vec{v} \neq \vec{0}$, then λ is an *eigenvalue* of A and $\vec{v}$ is a corresponding *eigenvector*. The zero vector and the set of all eigenvectors of A corresponding to λ constitute the *eigenspace* E_λ.

Note. By the Fundamental Theorem of Algebra, Theorem 9.1.A, every $n \times n$ complex matrix has n (not necessarily distinct) complex eigenvalues.

Examples. Page 485 Number 2(a), Page 485 Number 10(a).

Note. As with the definition of eigenvalues of eigenvectors, we have the same definitions of diagonalizable and similar in the complex setting as we had in the real setting.

Definition. An $n \times n$ complex matrix A is *diagonalizable* if there exists an invertible (complex) matrix C and a diagonal (complex) matrix D such that $D = C^{-1}AC$. Two $n \times n$ complex matrices A and B are *similar* if there is invertible (complex) matrix C such that $B = C^{-1}AC$.

Note. Some of the results of Section 5.2, "Diagonalization," hold for the complex setting. In particular, the following hold:

Theorem 5.2. Matrix Summary of Eigenvalues of A.

Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be (possibly complex) scalars and $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}$ be nonzero vectors in n-space. Let C be the $n \times n$ matrix having $\vec{v_j}$ as jth column vector and let

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Then $AC = CD$ if and only if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A and $\vec{v_j}$ is an eigenvector of A corresponding to λ_j for $j = 1, 2, \ldots, n$.

Corollary 1. A Criterion for Diagonalization.

An $n \times n$ matrix A is diagonalizable if and only if n-space has a basis consisting of eigenvectors of A.

Theorem 5.4. A Criterion for Diagonalization.

An $n \times n$ matrix A is diagonalizable if and only if the algebraic multiplicity of each (possibly complex) eigenvalue is equal to its geometric multiplicity.

Examples. Page 485 Number 2(b), Page 485 Number 10(b).

Note. We now term our attention to unitary matrices and show that a Hermitian matrix is diagonalizable by a unitary matrix.

Definition. Let A and B be $n \times n$ matrices. If there is a unitary matrix U such that $B = U^{-1}AU$ then A and B are *unitarily equivalent*.

Note. To prove our big result we need a preliminary result.

Theorem 9.4. Schur's Lemma.
Let A be an $n \times n$ (complex) matrix. There is a unitary matrix U such that $U^{-1}AU$ is upper triangular.

Note. Now for the main theorem of this section.

Theorem 9.5. The Spectral Theorem for Hermitian Matrices.
If A is a Hermitian matrix, then there exists a unitary matrix U such that $U^{-1}AU$ is a diagonal matrix (that is, A is *unitarily diagonalizable*). Furthermore, all eigenvalues of A are real.

Note. We can now prove what we stated as Theorem 5.5 in Section 5.2, "Diagonalization" (stated slightly differently here).

Corollary. Fundamental Theorem of Real Symmetric Matrices.
Every $n \times n$ real symmetric matrix has n real eigenvalues, counted with their algebraic multiplicity, and is diagonalizable by a real orthogonal matrix.

Note. Recall that Theorem 6.7, "Orthogonality of Eigenspaces of a Real Symmetric Matrix," states that eigenvectors of a real symmetric matrix that correspond to different eigenvalues are orthogonal. The next result shows that this also holds for Hermitian matrices.

Theorem 9.6. Orthogonality of Eigenspaces of a Hermitian Matrix.
The eigenvectors of a Hermitian matrix corresponding to distinct eigenvalues are orthogonal.

Examples. Page 485 Number 2(c), Page 485 Number 10(c).

Note. The following definition will be useful in classifying unitarily diagonalizable matrices.

Definition 9.5. A square matrix A is *normal* if it commutes with its conjugate transpose, $AA^* = A^*A$.

Theorem 9.7. Spectral Theorem for Normal Matrices.
A square matrix A is unitarily diagonalizable if and only if it is a normal matrix.

Note. A proof of Theorem 9.7 is to be given in Exercises 25 and 26.

Examples. Page 485 Number 20, Page 486 Number 22.

Revised: 5/8/2018