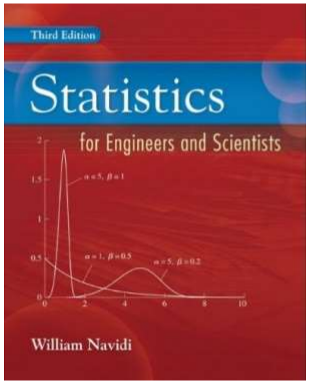
Foundations of Probability and Statistics-Calculus Based

Chapter 2. Probability

2.1. Basic Ideas—Exercises and Proofs of Theorems



Theorem 2.1.A

Theorem 2.1.A. For any event A, we have $P(A^c) = 1 - P(A)$. Also $P(\varnothing) = 0$.

Proof. Let $\mathcal{S}$ be the sample space and let A be an event. Then A and A^c are mutually exclusive, so by Axiom 3, $P(A \cup A^c) = P(A) + P(A^c)$. But $A \cup A^c = \mathcal{S}$, and be Axiom 1 we have $P(\mathcal{S}) = 1$. Therefore $P(A) + P(A^c) = P(A \cup A^c) = P(S) = 1$, so that $P(A^c) = 1 - P(A)$, as claimed.

Next, $\varnothing = \mathcal{S}^c$, and so by the first result,

$$P(\varnothing) = P(\mathcal{S}^c) = 1 - P(\mathcal{S}) = 1 - 1 = 0,$$

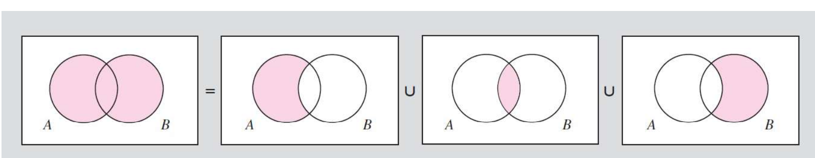
as claimed. □

Theorem 2.1.B

Theorem 2.1.B. Let A and B be any events. Then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof. First, we write A and B as a union of mutually exclusive events. We have $A \cup B = (A \cap B^c) \cup (A \cap B) \cup (A^c \cap B)$ (see the Venn diagram below).



By Axiom 3, $P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)$. Similarly, A and B can be written as mutually exclusive events, $A = (A \cap B^c) \cup (A \cap B)$ and $B = (A^c \cap B) \cup (A \cap B)$. Again by Axiom 3, $P(A) = P(A \cap B^c) + P(A \cap B)$ and $P(B) = P(A^c \cap B) + P(A \cap B)$.

Theorem 2.1.B (continued)

Theorem 2.1.B. Let A and B be any events. Then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof (continued). ... $P(A) = P(A \cap B^c) + P(A \cap B)$ and $P(B) = P(A^c \cap B) + P(A \cap B)$. Summing these we have

$$P(A) + P(B) = P(A \cap B^c) + P(A^c \cap B) + 2P(A \cap B)$$

$$= (P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)) + P(A \cap B).$$

Since $P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)$ as shown above, then $P(A) + P(B) = P(A \cup B) + P(A \cap B)$, or

$$P(A \cup B) = P(A) + P(B) - P(A \cap B),$$

as claimed. □