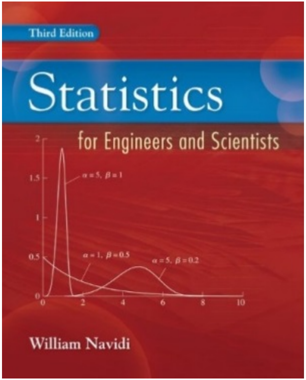
Foundations of Probability and Statistics-Calculus Based

Chapter 2. Probability

2.1. Basic Ideas—Exercises and Proofs of Theorems



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Proof. Let $\mathcal{S}$ be the sample space and let A be an event. Then A and A^c are mutually exclusive, so by Axiom 3, $P(A \cup A^c) = P(A) + P(A^c)$. But $A \cup A^c = \mathcal{S}$, and be Axiom 1 we have $P(\mathcal{S}) = 1$. Therefore $P(A) + P(A^c) = P(A \cup A^c) = P(S) = 1$, so that $P(A^c) = 1 - P(A)$, as claimed.

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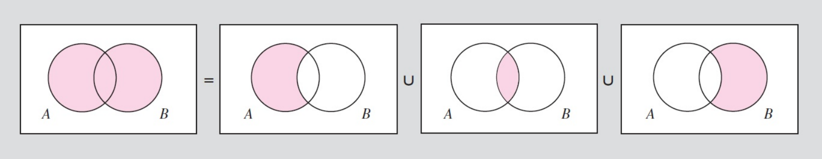

By Axiom 3, $P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)$. Similarly, A and B can be written as mutually exclusive events, $A = (A \cap B^c) \cup (A \cap B)$ and $B = (A^c \cap B) \cup (A \cap B)$. Again by Axiom 3, $P(A) = P(A \cap B^c) + P(A \cap B)$ and $P(B) = P(A^c \cap B) + P(A \cap B)$.

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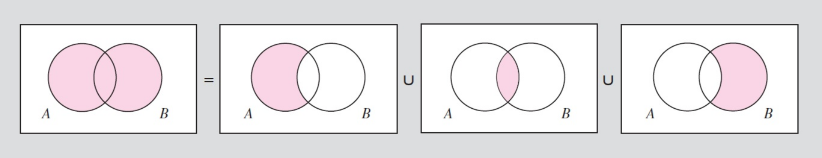

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Since $P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)$ as shown above, then $P(A) + P(B) = P(A \cup B) + P(A \cap B)$, or

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