Section 2.1. Basic Ideas.

Note. In this section we give basic definitions and informally introduce the idea of the probability of an event. We state and prove some properties of of the probability function.

Definition. An *experiment* is a process that results in an outcome that cannot be predicted in advance with certainty. The set of all possible outcomes of an experiment is the *sample space* for the experiment. A subset of a sample space is an *event*.

Note. Since we are starting with the basics, an example of an experiment is tossing a coin. The sample space is $\{\text{Heads}, \text{Tails}\}$. Another example of an experiment is rolling a 6-sided die. In this case, the sample space is $\{1, 2, 3, 4, 5, 6\}$ and an event is $\{2, 4, 6\}$ (the event of rolling an even number). An example of an experiment with a much larger sample space, is to choose a (real) number between 0 and 1 (inclusive, say). The sample space is then the numbers in the interval $[0, 1]$, or $\{x \in [0, 1] \mid x \in \mathbb{R}\}$. An event in this experiment is the interval $(1/4, 1/2) = \{x \in (1/4, 1/2) \mid x \in \mathbb{R}\}$ (the event of choosing a number between 1/4 and 1/2, excluding the endpoints). Notice that the empty set $\varnothing$ is an event in any experiment.

Example 2.1. (This example is a nod to the "Engineers" part of the title of the text book.) An electrical engineer has on hand two boxes of resistors, with four resistors in each box. The resistors in the first box are labeled 10 Ω (ohms), but in fact their resistances are 9, 10, 11, and 12 Ω. The resistors in the second box are labeled 20 Ω, but in fact their resistances are 18, 19, 20, and 21 Ω. The engineer chooses one resistor from each box and determines the resistance of each. Let A be the event that the first resistor has a resistance greater than 10, let B be the event that the second resistor has a resistance less than 19, and let C be the event that the sum of the resistances is equal to 28. Find a sample space for this experiment, and specify the subsets corresponding to the events A, B, and C.

Solution. Since the engineer chooses one resistor from each box, then we describe the chosen resistors with an ordered pair where the first entry is the resistance of the resistor chosen from the first box and the second entry is the resistance of the resistor chosen from the second box. Denoting the sample space as $\mathcal{S}$ we have:

$$\mathcal{S} = \{(9,18),(9,19),(9,20),(9,21),(10,18),(10,19),(10,20),(10,21),$$

$$(11,18),(11,19),(11,20),(11,21),(12,18),(12,19),(12,20),(12,21)\}.$$

For event A, we consider the subset of $\mathcal{S}$ where the first entry of a pair is greater than 10:

$$A = \{(11,18),(11,19),(11,20),(11,21),(12,18),(12,19),(12,20),(12,21)\}.$$

For event B, we consider the subset of $\mathcal{S}$ where the second entry of a pair is less than 19:

$$B = \{(9,18),(10,18),(11,18),(12,18)\}.$$

For event C, we consider the subset of $\mathcal{S}$ where the sum of the entries of a pair is equal to 28:

$$C = \{(9,19),(10,18)\}. \qquad \square$$

Note. We have define the sample space and an event of an experiment both as sets, so we quickly review some elementary properties of sets.

Definition. The *union* of two events A and B, denoted $A \cup B$, is the set of outcomes that belong to either A, to B, or to both. The *intersection* of events A and B, denoted $A \cap B$, is the set of outcomes that belong both to A and to B. The *complement* of an event A, denoted A^c, is the set of outcomes in the sample space that do not belong to A.

Note. We can use Venn diagrams to illustrate the previous definitions as:

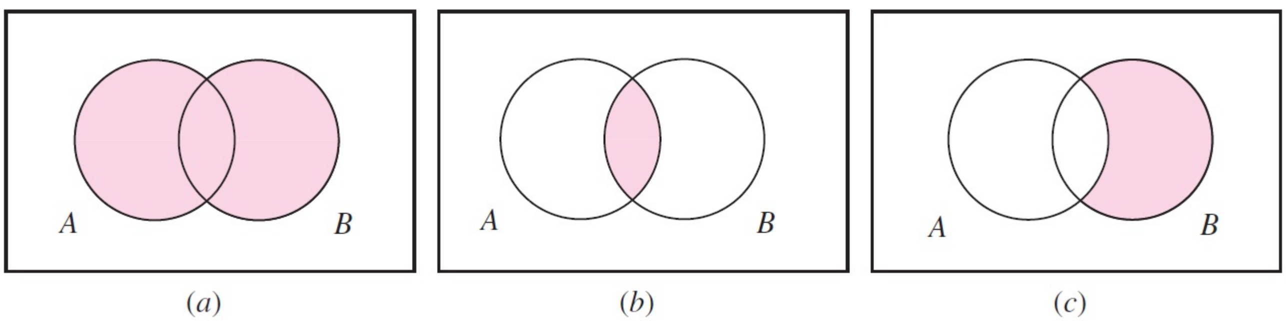

Figure 2.1. Venn diagrams illustrating various events:
(a) $A \cup B$, (b) $A \cap B$, (c) $B \cap A^c$.

Get used to the use of sets in probability and statistics! Notice that in Mathematical Statistics 1 (STAT 4047/5047; this is the next logical class to take after Foundations of Probability and Statistics—Calculus Based), after an introductory section, the first topic is Section 1.2. Sets.

Definition. Events A and B are *mutually exclusive* if they have no outcomes in common; that is, if $A \cap B = \varnothing$. More generally, a collection of events $A_1, A_2, \ldots, A_n$ is *mutually exclusive* if not two of the events have any outcomes in common; that is, if $A_i \cap A_j = \varnothing$ for all $i \neq j$.

Example 2.3. In Example 2.1, we had events

$$A = \{(11,18),(11,19),(11,20),(11,21),(12,18),(12,19),(12,20),(12,21)\}$$

$$B = \{(9,18),(10,18),(11,18),(12,18)\} \text{ and } C = \{(9,19),(10,18)\}.$$

Since $A \cap B = \{(11,18),(12,18)\} \neq \varnothing$, then events A and B are not mutually exclusive. Since $B \cap C = \{(11,18),(12,18)\} = C \neq \varnothing$, then events B and C are not mutually exclusive. However, $A \cap C = \varnothing$ and so events A and C are mutually exclusive. $\square$

Note. The plan is to associate a "probability" $P(A)$ with each event A associated with an experiment. Navidi states (see page 52): "$P(A)$ is the proportion of times that even A would occur in the long run, if the experiment were to be repeated over and over again." So Navidi is introducing probability with a frequency interpretation of probability (as defined in the Chapter 2. Introduction).

Axioms or Probability. We take the following three axioms:

Axiom 1. Let $\mathcal{S}$ be a sample space. Then $P(\mathcal{S}) = 1$.

Axiom 2. For any event A, $0 \leq P(A) \leq a$.

Axiom 3. If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

More generally, if $A_1, A_2, \ldots$ are mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots.$$

The first part of Axiom 3 which states $P(A \cup B) = P(A) + P(B)$ (for mutually exclusive A and B) is called *finite additivity*. The second part of Axiom 3 concerning mutually exclusive events $A_1, A_2, \ldots$ is called *countable additivity*.

Note 2.1.A. We have assumed that every subset of the sample space (that is, every event) has a probability associated with it. If the sample space is finite, then this should not be a problem. If it is "countable," we should also be okay. But if it is a set "bigger" than this, then we may have problems! Surprisingly, some infinite sets are bigger than others (you will explore this in Mathematical Reasoning [MATH 3000] and Analysis 1 [MATH 4217/5217; if interested, see my online notes for these classes on Mathematical Reasoning, notice Chapter 2. Sets, and on Analysis 1, notice Section 1-3. The Completeness Axiom). These concerns are dealt with (though not completely resolved) in Real Analysis 1 (MATH 5210) when defining Lebesgue measure which can be used to but a probability on many sets of real numbers (but not all, not even all bounded sets; see my notes on Section 2.6. Nonmeasurable Sets). We now happily carry on under the assumption that we won't encounter such weird stuff in this class.

Theorem 2.1.A. For any event A, we have $P(A^c) = 1 - P(A)$. Also $P(\varnothing) = 0$.

Exercise 2.1.8. An automobile insurance company divides customers into three categories, good risks, medium risks, and poor risks. Assume that 70% of the customers are good risks, 20% are medium risks, and 10% are poor risks. As part of an audit, one customer is chosen at random. **(a)** What is the probability that the customer is a good risk? **(b)** What is the probability that the customer is not a poor risk?

Solution. Let G, M, and R be the events that the chosen customer is a good risk, a medium risk, of a poor risk, respectively. Notice that the sample space is $\mathcal{S} = G \cup M \cup P$ and that events G, M, and R are mutually exclusive and that $P(G) = 0.70$, $P(M) = 0.20$, and $P(R) = 0.10$.
(a) The question is $P(G) =$? Well, as just observed, $\boxed{P(G) = 0.70}$.
(b) Since R denotes the event that the customer is a poor risk, the question is $P(R^c) =$? By Theorem 2.1.A, $P(R^c) = 1 - P(R)$, so $P(R^c) = 1 - (0.10) = \boxed{0.90}$ since $P(R) = 0.10$ as observed above. □

Note 2.1.B. If event A contains (distinct) outcomes $O_1, O_2, \ldots, O_n$, then we can treat set A as a union of mutually disjoint sets containing one outcome (a set with only one element if often called a *singleton*) then $A = \{O_1\} \cup \{O_2\} \cup \cdots \cup \{O_n\}$ and Axiom 3 $P(A) = P(O_1) + P(O_2) + \cdots + P(O_n)$. A finite population from which an outcome is sampled at random can be thought of as a sample space with equally likely outcomes. So if the sample space $\mathcal{S}$ contains N equally likely outcomes, and it A is an event containing k outcomes, then $P(A) = k/N$. We can use this idea to find probabilities empirically if we know the population.

Theorem 2.1.B. Let A and B be any events. Then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Exercise 2.1.14. Inspector A visually inspected 1000 ceramic bowls for surface flaws and found flaws in 37 of them. Inspector B inspected the same bowls and found flaws in 43 of them. A total of 948 bowls were found to be flawless by both inspectors. One of the 1000 bowls is selected at random. **(a)** Find the probability that a flaw was found in this bowl by at least one of the two inspectors. **(b)** Find the probability that flaws were found in this bowl by both inspectors. **(c)** Find the probability that a flaw was found by Inspector A but not by Inspector B.

Solution. We let A denote the set of bowls for which Inspector A finds flaws, and let B denote the set of bowls for which Inspector B finds flaws. Based on Note 2.1.B and the data given, we say that $P(A) = 37/1000$ and $P(B) = 43/1000$. Since 948 bowls were found to be flawless by both inspectors, then $1000 - 948 = 52$ bowls were found to be flawed by either Inspector A or Inspector B. The bowls that were found to be flawed by either Inspector A or Inspector B make up the event $A \cup B$, so that $P(A \cup B) = 52/1000$.

(a) The bowls in $A \cup B$ are those found to be flawed by at least on of the two inspectors (that is, the bowls which either A or B found flawed). So we are interested in the probability that the one bowl chosen is in $A \cup B$. This is, as described above, $P(A \cup B) = \boxed{52/1000}$.

(b) The bowls that were found to be flawed by both inspectors make up event $A \cap B$. We know that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ by Theorem 2.1.B, so we have

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 37/1000 + 43/1000 - 52/1000 = \boxed{28/1000}.$$

(c) The bowls that were found to be flawed by Inspector A but not by Inspector B make up the event $A \cap B^c$. Notice that we can write event A as the union of two mutually exclusive events: $A = (A \cap B) \cup (A \cap B^c)$. This follows from the Venn diagram:

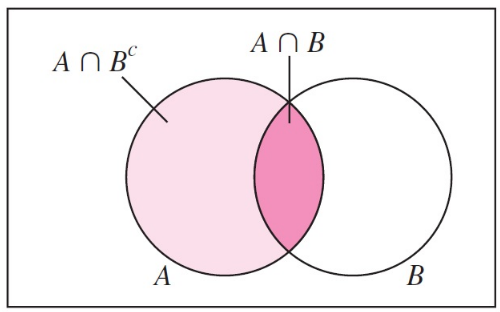

Figure 2.3

So by Axiom 3, $P(A) = P(A \cap B) + P(A \cap B^c)$, or $P(A \cap B^c) = P(A) - P(A \cap B)$. Since $P(A) = 37/1000$ as described above, and $P(A \cap B) = 28/1000$ by part (b), then $P(A \cap B^c) = 37/1000 - 28/1000 = \boxed{9/1000}$. □

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