Section 2.2. Counting Methods.

Note. In this section we state the Fundamental Principle of Counting and use it to count permutations (where order matters) and combinations (where order does not matter).

Example. Consider the experiment of rolling a six-sided die. The sample space is a set of 6 outcomes: $\{1,2,3,4,5,6\}$. The experiment of rolling a six-sided die a first time and a second time results in the sample space (treating the outcomes as ordered pairs) of 3 outcomes:

$$\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),$$

$$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$$

$$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}.$$

This is suggestive of a special case of the Fundamental Principle of Counting:

> If an operation can be performed in n_1 ways, and if for each of these ways a second operation can be performed in n_2 ways, then the total number of ways to perform the two operations is $n_1 n_2$.

We could roll the six-sided die a third time and we would not be surprised to learn that the sample space consists of $6^3 = 216$ outcomes. This hints at the more general Fundamental Principle of Counting, stated next.

Note 2.2.A. The **Fundamental Principle of Counting** states:

> Assume that k operations are to be performed. If there are n_1 ways to perform the first operation, and if for each of these ways there are n_2 ways to perform the second operation, and if for each choice of ways to perform the first two operations there are n_3 ways to perform the third operation, and so on, then the total number of ways to perform the sequence of k operations is $n_1 n_2 \cdots n_k$.

Exercise 2.2.2. A chemical engineer is designing an experiment to determine the effect of temperature, stirring rate, and type of catalyst on the yield of a certain reaction. She wants to study five different reaction temperatures, two different stirring rates, and four different catalysts. If each run of the experiment involves a choice of one temperature, one stirring rate, and one catalyst, how many different runs are possible?

Solution. In the notation of the Fundamental Principle of Counting, we have $k = 3$ operations to be performed. There are $n_1 = 5$ ways to perform the first operation (reaction temperature), $n_2 = 2$ ways to perform the second operation (stirring rates), and $n_3 = 4$ ways to perform the third operation (catalyst). So by the Fundamental Principle of Counting, there are $n_1 n_2 n_3 = (5)(2)(4) = \boxed{40}$ different possible runs of the experiment. □

Definition. A *permutation* is an ordering of a finite collection of objects.

Note 2.2.B. Consider a collection of n objects. We can use the Fundamental Principle of Counting to count the number of permutations of these objects. There are n choices for the first object, $n-1$ choices for the second object, $\ldots$, $n-i+1$ choices for the ith object, $\ldots$, 2 choices for the $(n-1)$th object, and 1 choice for the nth object. So the number of permutations of this collection of n objects is $n(n-1)(n-2)\cdots(3)(2)(1)$. This quantity is called "n factorial" and denoted $n!$.

Definition. For positive integer n, define n *factorial* as

$$n! = n(n-1)(n-2)\cdots(3)(2)(1).$$

Define $0! = 1$.

Example 2.12. Five people stand in line at a movie theater. Into how many different orders can they be arranged?

Solution. This is simply a permutation of 5 objects, so the number of "arrangements" is $5! = (5)(4)(3)(2)(1) = \boxed{120}$. □

Note 2.2.C. Suppose we are interested in a permutation of k objects chosen from a collection of n objects, where $k \leq n$. Similar to the argument given for a number of permutations, there are n choices for the first object, $n-1$ choices for the second object, $\ldots$, $n-k+1$ choices for the kth object. So by the Fundamental Principle of Counting, $(n)(n-1)\cdots(n-k+1)$ ways to make these arrangements of k objects chosen from a collection of n object. That is, the number of permutations of k

objects chosen from a group of n objects is

$$(n)(n-1)\cdots(n-k+1) = \frac{(n)(n-1)\cdots(n-k+1)(n-k)(n-k-1)\cdots(3)(2)(1)}{(n-k)(n-k-1)\cdots(3)(2)(1)} = \frac{n!}{(n-k)!}.$$

Definition. Consider a set of n objects. A subset of size k of the set of n objects is a *combination.*

Note 2.2.D. We now count the number of combinations of k objects from a set of size n. We know the number of ways to arrange k objects chosen from the set of n object is (by Note 2.2.C) $\frac{n!}{(n-k)!}$. But in a combination, we are not interested in the ordering of the objects. So we consider how many ways there are to arrange a given set of k objects. This is just the number of permutations of k objects which, by Note 2.2.B, is $k!$. So in $\frac{n!}{(n-k)!}$, each subset has been counted $k!$ times. Hence the number of combinations of k objects chosen from a group of n objects is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Example 2.14. At a certain event, 30 people attend, and 5 will be chosen at random to receive door prizes. The prizes are all the same, so the order in which the people are chosen does not matter. How many different groups of five people can be chosen?

Solution. Since order does not matter, we are interested in the number of combinations of $k = 5$ objects chosen from a group of $n = 30$. So the number of different groups of five people is

$$\binom{30}{5} = \frac{30!}{5!(30-5)!} = \frac{30!}{5!25!} = \frac{(30)(29)(28)(27)(26)}{(5)(4)(3)(2)(1)} = \frac{17{,}100{,}720}{120} = \boxed{142{,}506}. \quad \square$$

Note 2.2.E. Suppose we want to break a group of n objects into groups of $k_1, k_2, \ldots, k_r$ objects where $k_1 + k_2 + \cdots + k_r = n$. By Note 2.2.D, there are $\binom{n}{k_1} = \frac{n!}{k_1!(n-k_1)!}$ ways to choose the objects in the first group. Once this is done, there are now $n - k_1$ objects left and so there are $\binom{n-k_1}{k_2} = \frac{(n-k_1)!}{k_2!(n-k_1-k_2)!}$ ways to choose the objects in the second group. Similarly, there are

$$\binom{n-k_1-k_2-\cdots-k_{i-1}}{k_i} = \frac{(n-k_1-k_2-\cdots-k_{i-1})!}{k_i!(n-k_1-k_2-\cdots-k_i)!}$$

ways to choose the objects in the ith group, where $1 \le i \le r-1$. Now after the first $r-1$ groups are chosen, then the rth group is determined automatically (or, since $n-k_1-k_2-\cdots k_{r-1} = k_r$, there are $\binom{k_r}{k_r} = 1$ way to choose the rth group). So by the Fundamental Principle of Counting, the number of ways to break the group of n objects into the r groups of sizes $k_1, k_2, \ldots, k_r$ (where $k_1 + k_2 + \cdots + k_r = n$) is:

$$\left(\frac{n!}{k_1!(n-k_1)!}\right)\left(\frac{(n-k_1)!}{k_2!(n-k_1-k_2)!}\right)\cdots\left(\frac{(n-k_1-k_2-\cdots-k_{r-2})!}{k_{r-1}!(n-k_1-k_2-\cdots-k_{r-1})!}\right)\left(\frac{k_r!}{k_r!}\right) =$$

$$= \frac{n!}{k_1!k_2!\cdots k_r!}.$$

Exercise 2.2.10. A company has hired 15 new employees, and must assign 6 to the day shift, 5 to the graveyard shift, and 4 to the night shift. In how many ways can the assignment be made?

Solution. In the notation of Note 2.2.E, we have a collection of $n = 15$ objects and want to break them into $r = 3$ groups, where the groups have sizes $k_1 = 6$, $k_2 = 5$, and $k_3 = 4$. The number of ways to do this is (by Note 2.2.E),

$$\frac{n!}{k_1!k_2!\cdots k_r!} = \frac{15!}{6!5!4!} = \frac{(15)(14)(13)(12)(11)(10)(9)(8)(7)}{(120)(24)}$$

$$= \frac{1{,}816{,}214{,}400}{2{,}880} = \boxed{630{,}630}. \ \square$$

Exercise 2.2.12. A drawer contains 6 red socks, 4 green socks, and 2 black socks. Two socks are chosen at random. What is the probability that they match?

Solution. There is a total of $n = 12$ socks, and so by Note 2.2.D, the number of ways to choose $k = 2$ socks is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{12!}{2!(12-2)!} = \frac{12!}{2!10!} = \frac{(12)(11)}{2} = 66.$$

Now the event that the two socks match is made up of the mutually exclusive events of choosing two red socks, choosing two green socks, and choosing 2 black socks. The number of ways to choose two red socks is (again by Note 2.2.D) $\dfrac{6!}{2!(6-2)!} = \dfrac{6!}{2!4!} = \dfrac{(6)(5)}{(2)} = 15$, the number of ways to choose two green socks is $\dfrac{4!}{2!(4-2)!} = \dfrac{4!}{2!2!} = \dfrac{(4)(3)}{(2)} = 6$, and the number of ways to choose two black socks is $\dfrac{2!}{2!(2-2)!} = \dfrac{2!}{2!0!} = 1$. We then have (see Note 2.1.B) that the probability of the socks matching is

$$\frac{15+6+1}{66} = \frac{22}{66} = \boxed{\frac{1}{3}}. \ \square$$