Chapter 15. Multiple Integrals
15.2. Double Integrals over General Regions—Examples and Proofs of Theorems
Exercise 15.2.20

Exercise 15.2.20. Sketch the region of integration and evaluate the double integral \( \int_0^\pi \int_0^{\sin x} y \, dy \, dx \).

Solution. The region is:
Exercise 15.2.20. Sketch the region of integration and evaluate the double integral \( \int_0^\pi \int_0^{\sin x} y \, dy \, dx \).

Solution. The region is:

We evaluate the iterated integral as:

\[
\int_0^\pi \int_0^{\sin x} y \, dy \, dx = \int_0^\pi \left. \frac{y^2}{2} \right|_{y=0}^{y=\sin x} dx = \int_0^\pi \frac{\sin^2 x}{2} \, dx
\]
Exercise 15.2.20. Sketch the region of integration and evaluate the double integral \( \int_0^\pi \int_0^{\sin x} y \, dy \, dx \).

Solution. The region is:

\[
\begin{aligned}
&\int_0^\pi \int_0^{\sin x} y \, dy \, dx = \int_0^\pi \left. \frac{y^2}{2} \right|_{y=\sin x} \, dx = \int_0^\pi \frac{\sin^2 x}{2} \, dx - 0 \, dx
\end{aligned}
\]
Exercise 15.2.20. Sketch the region of integration and evaluate the double integral \[ \int_0^\pi \int_0^{\sin x} y \, dy \, dx. \]

Solution (continued).

\[ = \int_0^\pi \frac{1}{2} \frac{1 - \cos 2x}{2} \, dx \quad \text{since} \quad \sin^2 x = \frac{1 - \cos 2x}{2} \]

\[ = \frac{x}{4} - \frac{\sin 2x}{8} \bigg|_{x=0}^{x=\pi} = \left( \frac{\pi}{4} - \frac{\sin 2\pi}{8} \right) - (0) = \frac{\pi}{4}. \]
Exercise 15.2.40. Sketch the region of integration and write the equivalent double integral with the order of integration reverse:
\[ \int_0^2 \int_0^{4-y^2} y \, dx \, dy. \]

Solution. We first have \( x \) ranging from 0 to \( 4 - y^2 \), and second \( y \) ranges from 0 to 2. So the region is:
Exercise 15.2.40. Sketch the region of integration and write the equivalent double integral with the order of integration reverse:
\[ \int_{\frac{4-y^2}{2}}^{2} \int_{0}^{4-y^2} y \, dx \, dy. \]

**Solution.** We first have \( x \) ranging from 0 to \( 4-y^2 \), and second \( y \) ranges from 0 to 2. So the region is:

Now we can interpret that first \( y \) ranges from 0 to the curve \( x = 4-y^2 \) (or \( y = \sqrt{4-x} \), since \( y \geq 0 \) on the region) and second \( x \) ranges from 0 to 4. So the integral becomes
\[ \int_{0}^{4} \int_{0}^{\sqrt{4-x}} y \, dy \, dx. \]
Exercise 15.2.40. Sketch the region of integration and write the equivalent double integral with the order of integration reverse:

$$\int_0^2 \int_0^{4-y^2} y \, dx \, dy.$$

**Solution.** We first have $x$ ranging from 0 to $4 - y^2$, and second $y$ ranges from 0 to 2. So the region is:

Now we can interpret that first $y$ ranges from 0 to the curve $x = 4 - y^2$ (or $y = \sqrt{4-x}$, since $y \geq 0$ on the region) and second $x$ ranges from 0 to 4. So the integral becomes

$$\int_0^4 \int_0^{\sqrt{4-x}} y \, dy \, dx.$$
Exercise 15.2.50

**Exercise 15.2.50.** Sketch the region of integration, reverse the order of integration, and evaluate the integral:

\[
\int_{0}^{2} \int_{0}^{4-x^2} \frac{xe^{2y}}{4-y} \, dy \, dx.
\]

**Solution.** We first have \( y \) ranging from 0 to \( 4-x^2 \), and second \( x \) ranges from 0 to 2. So the region is:
Exercise 15.2.50. Sketch the region of integration, reverse the order of integration, and evaluate the integral:

\[ \int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} \, dy \, dx. \]

Solution. We first have \( y \) ranging from 0 to \( 4 - x^2 \), and second \( x \) ranges from 0 to 2. So the region is:

Now we can interpret that first \( x \) ranges from 0 to the curve \( y = 4 - x^2 \) (or \( x = \sqrt{4 - y} \), since \( x \geq 0 \) on the region) and second \( y \) ranges from 0 to 4. So the integral becomes

\[ \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y} \, dx \, dy. \]
Exercise 15.2.50. Sketch the region of integration, reverse the order of integration, and evaluate the integral:

$$
\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} \, dy \, dx.
$$

Solution. We first have $y$ ranging from 0 to $4 - x^2$, and second $x$ ranges from 0 to 2. So the region is:

Now we can interpret that first $x$ ranges from 0 to the curve $y = 4 - x^2$ (or $x = \sqrt{4 - y}$, since $x \geq 0$ on the region) and second $y$ ranges from 0 to 4. So the integral becomes

$$
\int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y} \, dx \, dy.
$$
Exercise 15.2.50 (continued)

Exercise 15.2.50. Sketch the region of integration, reverse the order of integration, and evaluate the integral:

\[ \int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} \, dy \, dx. \]

Solution (continued). We now evaluate the new iterated integral:

\[ \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y} \, dx \, dy = \int_0^4 \frac{x^2 e^{2y}}{2(4-y)} \bigg|_{x=0}^{x=\sqrt{4-y}} \, dy \]

\[ = \int_0^4 \frac{(\sqrt{4-y})^2 e^{2y}}{2(4-y)} - 0 \, dy = \int_0^4 \frac{(4-y)e^{2y}}{2(4-y)} \, dy = \int_0^4 \frac{e^{2y}}{2} \, dy \]

\[ = \frac{e^{2y}}{4} \bigg|_{y=0}^{y=4} = \frac{e^{2(4)}}{4} - \frac{e^{2(0)}}{4} = \frac{e^8 - 1}{4}. \]
Exercise 15.2.58. Find the volume of the solid that is bounded above the cylinder \( z = x^2 \) and below by the region enclosed by the parabola \( y = 2 - x^2 \) and the line \( y = x \) in the xy-plane.

Solution. The region \( R \) is:
Exercise 15.2.58. Find the volume of the solid that is bounded above the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the $xy$-plane.

Solution. The region $R$ is:

First $y$ ranges from $x$ to $2 - x^2$, and second $x$ ranges from $-2$ to $1$. Since $z = f(x, y) = x^2$ is nonnegative over $R$ then the desired volume is

$$V = \int_{-2}^{1} \int_{x}^{\sqrt{2-x^2}} x^2 \, dy \, dx.$$
Exercise 15.2.58. Find the volume of the solid that is bounded above the cylinder \( z = x^2 \) and below by the region enclosed by the parabola \( y = 2 - x^2 \) and the line \( y = x \) in the \( xy \)-plane.

Solution. The region \( R \) is:

First \( y \) ranges from \( x \) to \( 2 - x^2 \), and second \( x \) ranges from \( -2 \) to \( 1 \).

Since \( z = f(x, y) = x^2 \) is nonnegative over \( R \) then the desired volume is

\[
V = \int_{-2}^{1} \int_{x}^{\sqrt{2-x^2}} x^2 \, dy \, dx.
\]
Solution (continued). So the volume is

\[
V = \int_{-2}^{1} \int_{y}^{\sqrt{2-x^2}} x^2 \, dy \, dx = \int_{-2}^{1} x^2 y \bigg|_{y=x^2} dx
\]

\[
= \int_{-2}^{1} x^2 (2-x^2) - x^2(x) \, dx = \int_{-2}^{1} 2x^2 - x^4 - x^3 \, dx = \frac{2x^3}{3} - \frac{x^5}{5} - \frac{x^4}{4} \bigg|_{x=-2}^{x=1}
\]

\[
= \left( \frac{2(1)^3}{3} - \frac{(1)^5}{5} - \frac{(1)^4}{4} \right) - \left( \frac{2(-2)^3}{3} - \frac{(-2)^5}{5} - \frac{(-2)^4}{4} \right)
\]

\[
= \frac{2}{3} - \frac{1}{5} - \frac{1}{4} + \frac{16}{3} - \frac{32}{5} + 4 = \frac{40}{60} - \frac{12}{60} - \frac{15}{60} + \frac{320}{60} - \frac{384}{60} + \frac{240}{60} = \frac{189}{60} = \frac{63}{20}.
\]
Exercise 15.2.76

Exercise 15.2.76. (Unbounded Region) Integrate

\[ f(x, y) = \frac{1}{(x^2 - x)(y - 1)^{2/3}} \]

over the infinite rectangle \( 2 \leq x < \infty, \ 0 \leq y \leq 2 \).

Solution. We want to find

\[ \int_{2}^{\infty} \int_{0}^{2} \frac{1}{(x^2 - x)(y - 1)^{2/3}} \, dy \, dx. \]

This is an improper integral and so we write it as a limit:

\[ \int_{2}^{\infty} \int_{0}^{2} \frac{1}{(x^2 - x)(y - 1)^{2/3}} \, dy \, dx = \lim_{b \to \infty} \int_{2}^{b} \int_{0}^{2} \frac{1}{(x^2 - x)(y - 1)^{2/3}} \, dy \, dx \]

\[ = \lim_{b \to \infty} \int_{2}^{b} \frac{1}{x^2 - x} \left( (y - 1)^{1/3} \right)_{y=0}^{y=2} \, dx \]

\[ = \lim_{b \to \infty} \int_{2}^{b} \frac{1}{x^2 - x} \left( 3((2) - 1)^{1/3} - \frac{1}{x^2 - x} 3((0) - 1)^{1/3} \right) dx \]
Exercise 15.2.76

Exercise 15.2.76. (Unbounded Region) Integrate
\[ f(x, y) = \frac{1}{(x^2 - x)(y - 1)^{2/3}} \] over the infinite rectangle \( 2 \leq x < \infty, 0 \leq y \leq 2 \).

Solution. We want to find
\[ \int_2^\infty \int_0^2 \frac{1}{(x^2 - x)(y - 1)^{2/3}} \, dy \, dx. \] This is an improper integral and so we write it as a limit:

\[ \int_2^\infty \int_0^2 \frac{1}{(x^2 - x)(y - 1)^{2/3}} \, dy \, dx = \lim_{b \to \infty} \int_2^b \int_0^2 \frac{1}{(x^2 - x)(y - 1)^{2/3}} \, dy \, dx \]

\[ = \lim_{b \to \infty} \int_2^b \frac{1}{x^2 - x} \, \frac{(y - 1)^{1/3}}{1/3} \bigg|_{y=0}^{y=2} \, dx \]

\[ = \lim_{b \to \infty} \int_2^b \frac{1}{x^2 - x} \left( 3((2) - 1)^{1/3} - \frac{1}{x^2 - x} \right) 3((0) - 1)^{1/3} \, dx \]
Exercise 15.2.76. (Unbounded Region) Integrate
\[ f(x, y) = \frac{1}{(x^2 - x)(y - 1)^{2/3}} \]
over the infinite rectangle \( 2 \leq x < \infty \), \( 0 \leq y \leq 2 \).

Solution (continued).

\[
\begin{align*}
&= \lim_{b \to \infty} \int_{2}^{b} \frac{6}{x^2 - x} \, dx = 6 \lim_{b \to \infty} \int_{2}^{b} \frac{1}{x - 1} - \frac{1}{x} \, dx \text{ by partial fractions} \\
&= 6 \lim_{b \to \infty} \left( \ln(x - 1) - \ln x \right) \bigg|_{x=2}^{x=b} = 6 \lim_{b \to \infty} \ln \left( \frac{x - 1}{x} \right) \bigg|_{x=2}^{x=b} \\
&= 6 \lim_{b \to \infty} \ln \left( \frac{b - 1}{b} \right) - 6 \ln \left( \frac{2 - 1}{2} \right) = -6 \ln(1/2) = 6 \ln 2.
\end{align*}
\]