Calculus 3

Chapter 15. Multiple Integrals

15.3. Area by Double Integration—Examples and Proofs of Theorems

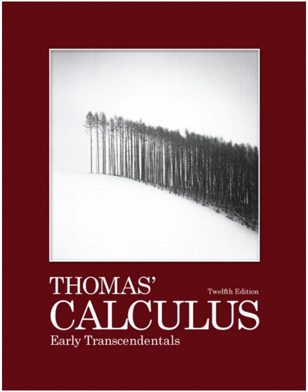

Table of contents

Exercise 15.3.8

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Solution. Notice the parabolas intersect when $y^2 - 1 = 2y^2 - 2$ or $y^2 = 1$ or $y = \pm 1$ (and $x = 0$). The region is:

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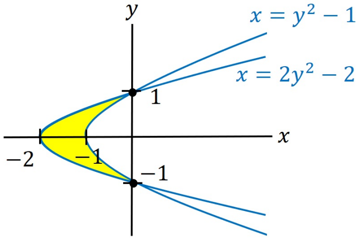

So with a dy-slice, we have x ranging from $x = 2y^2 - 2$ to $x = y^2 - 1$. Then y ranges from -1 to 1.

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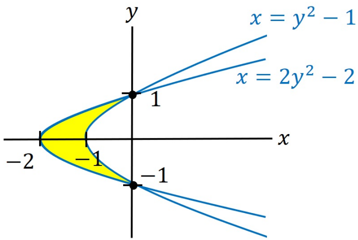

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Solution (continued). So the area is:

$$\iint_R 1\, da = \int_{-1}^{1} \int_{x=2y^2-2}^{x=y^2-1} 1\, dx\, dy = \int_{-1}^{1} \left(x \Big|_{x=2y^2-2}^{x=y^2-1} \right) dy$$

$$= \int_{-1}^{1} ((y^2-1) - (2y^2-2))\, dy = \int_{-1}^{1} (-y^2+1)\, dy = \left(\frac{-1}{3} y^3 + y \right) \Big|_{-1}^{1}$$

$$= \left(\frac{-1}{3}(1)^3 + (1) \right) - \left(\frac{-1}{2}(-1) + (-1) \right) \frac{4}{3}.$$

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$$= \int_{-1}^{1}((y^2-1)-(2y^2-2))\,dy = \int_{-1}^{1}(-y^2+1)\,dy = \left(\frac{-1}{3}y^3+y\right)\Bigg|_{-1}^{1}$$

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Exercise 15.3.14. Consider $\int_0^3 \int_{-x}^{x(2-x)} dy\, dx$. This represents the area of a region in the xy-plane. Sketch the region, label each bounding curve with its equation, and give the coordinates of the points where the curves intersect. Then find the area of the region.

Solution. The curve $y = -x$ is a line through the origin with slope $m = -1$. The curve $y = x(2-x) = 2x - x^2$ is a concave down parabola with vertes at $(1, 1)$. The curves intersect when $-x = 2x - x^2$ or $x^2 - 3x = 0$ or $x = 0$ and $x = 3$. The region is then:

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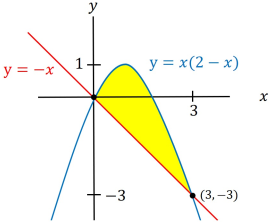

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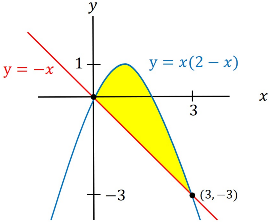

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Solution (continued). So with a dx-slice, we have y ranging from $y = -x$ to $y = x(2-x)$. Then x ranges from 0 to -3. So the area is

$$A = \iint_R 1\, dA = \int_0^3 \int_{-x}^{2x-x^2} 1\, dy\, dx = \int_0^3 \left(y \Big|_{y=-x}^{y=2x-x^2} \right) dx$$

$$\int_0^3 ((2x - x^2) - (-x))\, dx = \int_0^3 (3x - x^2)\, dx = \left(\frac{3}{2}x^2 - \frac{1}{3} \right) \Big|_0^3$$

$$= \left(\frac{3}{2}(3)^2 - \frac{1}{3}(3)^3 \right) - \left(\frac{3}{2}(0)^2 - \frac{1}{3}(0)^3 \right)$$

$$= (3/2)(0) - (1/3)(27) - 0 = (27/2) - 9 = 9/2. \quad \square$$

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Exercise 15.3.20. Calculate the average value of $f(x,y) = xy$ over the square $0 \leq x \leq 1$, $0 \leq y \leq 1$ and over the quarter circle $x^2 + y^2 \leq 1$ in the first quadrant.

Solution. Over the square R_1 we have:

$$\left(\begin{array}{c} \text{Average Value} \\ \text{of } f \text{ over } R_1 \end{array} \right) = \frac{1}{\text{area of } R_1} \iint_{R_1} f(x,y)\, dA = \frac{1}{(1)(1)} \int_0^1 \int_0^1 xy\, dx\, dy$$

$$= \int_0^1 \left(\frac{1}{2} x^2 y \right) \Big|_{x=0}^{x=1} dy = \int_0^1 \frac{1}{2} y\, dy = \frac{1}{4} y^2 \Big|_0^1 = \frac{1}{4}.$$

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We write the circle as $x = \sqrt{1-y^2}$, let x range from $x = 0$ to $x = \sqrt{1-y^2}$ and then let y range form 0 to 1. Then over the quarter circle R_2 we have

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Solution (continued). ...

$$\left(\begin{array}{c} \text{Average Value} \\ \text{of } f \text{ over } R_2 \end{array} \right) = \frac{1}{\pi(1)^2/4} \int_0^1 \int_0^{\sqrt{1-y^2}} xy \, dx \, dy$$

$$= \frac{4}{\pi} \int_0^1 \left(\frac{1}{2} x^2 y \right) \Bigg|_{x=0}^{x=\sqrt{1-y^2}} dy = \sqrt{4}\pi \int_0^1 \frac{1}{2} (\sqrt{1-y^2})^2 y \, dy$$

$$= \frac{2}{\pi} \int_0^1 (y - y^3) \, dy = \frac{2}{\pi} \left(\frac{1}{2} y^2 - \frac{1}{4} y^4 \right) \Bigg|_0^1 = \frac{2}{\pi} \left(\frac{1}{2}(1)^2 - \frac{1}{r}(1)^4 \right) = 0 = \frac{1}{2\pi}.$$

□