Chapter 15. Multiple Integrals
15.4. Double Integrals in Polar Form—Examples and Proofs of Theorems

Exercise 15.4.6

Exercise 15.4.6. Describe the given region in polar coordinates:

Solution. Since \( x = r \cos \theta \) then along the vertical line \( x = 1 \) we have \( 1 = r \cos \theta \) or \( r = 1/\cos \theta = \sec \theta \). Along the circle we have \( r = 2 \). So we can describe the region in terms of \( r \)-limits. A typical ray from the origin enters the region where \( r = \sec \theta \) and leaves where \( r = 2 \). To find \( \theta \)-limits, notice the following triangle:

Exercise 15.4.10

Exercise 15.4.10. Change the integral into an equivalent polar integral. Then evaluate the polar integral:

\[
\int_{0}^{1} \int_{0}^{\sqrt{1-y^2}} (x^2 + y^2) \, dx \, dy.
\]

Solution. With \( x = \sqrt{1-y^2} \) we have \( x^2 + y^2 = 1 \) where \( x \geq 0 \) and \( x^2 + y^2 = 1 \) for \( y \geq 0 \). This is the upper half of the unit circle centered at the origin. With \( y \) ranging from 0 to 1 we then have the region:
Exercise 15.4.10 (continued)

$$x^2 + y^2 = 1, x \geq 0$$

**Solution (continued).** We can take the $r$-limits as 0 to 1 and the $\theta$-limits as 0 to $\pi$. Since $r^2 = x^2 + y^2$, the integral becomes

$$
\int_0^1 \int_0^{\sqrt{1-r^2}} (x^2 + y^2) \, dx \, dy = \int_0^\pi \int_0^1 r^2 \, r \, dr \, d\theta = \int_0^\pi \frac{1}{4} r^4 \Bigg|_0^1 \, d\theta = \int_0^\pi \frac{1}{4} \, d\theta = \frac{1}{4} \theta \Bigg|_0^\pi = \frac{\pi}{4}.
$$

Exercise 15.4.28

**Exercise 15.4.28.** Find the area of the region that lies inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 1$.

**Solution.** The graphs of the cardioid (which is Exercise #1 on page 652 in Section 11.4) and the circle are:

So the $r$-limits of the region are 1 and $1 + \cos \theta$ and the $\theta$-limits are $-\pi/2$ and $\pi/2$.

Exercise 15.4.28 (continued)

**Solution (continued).** So the area of the region is

$$A = \int_{-\pi/2}^{\pi/2} r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \left( \frac{1}{2} (1 + \cos \theta)^2 - \frac{1}{2} \right) \, d\theta = \int_{-\pi/2}^{\pi/2} \left( \cos \theta + \frac{1}{2} \cos^2 \theta \right) \, d\theta = \sin \theta + \frac{1}{2} \left( \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) \bigg|_{\theta=-\pi/2}^{\theta=\pi/2} \text{ since } \int \cos^2 x \, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C,$$

by Example 9 in Section 5.5

$$= \left( \sin(\pi/2) + \frac{1}{2} \left( \frac{\pi}{2} + \sin 2(\pi/2) \right) \right) - \left( \sin(-\pi/2) + \frac{1}{2} \left( \frac{-\pi}{2} + \sin 2(-\pi/2) \right) \right) = (1 + (1/2)(\pi/4 + 0)) - (1 + (1/2)(-\pi/4 + 0)) = 2 + \pi/4. \qed $$

Exercise 15.4.38

**Exercise 15.4.38.** Converting to a Polar Integral. Integrate $f(x, y) = \frac{\ln(x^2 + y^2)}{x^2 + y^2}$ over the region $1 \leq x^2 + y^2 \leq e^2$.

**Solution.** We have $r^2 = x^2 + y^2$ so $\frac{\ln(x^2 + y^2)}{x^2 + y^2} = \frac{\ln r}{r}$. The region is an annulus with inner radius 1 and outer radius $e^2$:
Exercise 15.4.38 (continued)

**Solution.** We describe this with \( r \)-limits of 1 and \( e^2 \), and \( \theta \)-limits of 0 and \( 2\pi \). Since \( r^2 = x^2 + y^2 \) and \( dx
dy = r dr d\theta \), the integral is then:

\[
\iint_R \frac{\ln(x^2 + y^2)}{x^2 + y^2} \, dx \, dy = \int_0^{2\pi} \int_1^{e^2} \frac{\ln r}{r} \, r \, dr \, d\theta = \int_0^{2\pi} \int_1^{e^2} \ln r \, d\theta
\]

\[
= \left[ \int_1^{e^2} (f \ln r - r) \right]_{r=1}^{r=e^2} \, d\theta \text{ since } \int \ln x \, dx = x \ln x - x + C
\]

by Example 2 in Section 8.1 on page 456

\[
= \int_0^{2\pi} ((e^2 \ln e^2 - e^2)(1 \ln 1 - 1)) \, d\theta = \int_0^{2\pi} ((2e^2 - e^2) - (0 - 1)) \, d\theta
\]

\[
= \int_0^{2\pi} (e^2 + 1) \, d\theta = (e^2 + 1) \theta \bigg|_0^{2\pi} = 2\pi(e^2 + 1). \quad \square
\]

Exercise 15.4.41

**Exercise 15.4.41. Converting to Polar Integrals.**

(a) The usual way to evaluate the improper integral \( I = \int_0^\infty e^{-x^2} \, dx \) is first to calculate its square:

\[
I^2 = \left( \int_0^\infty e^{-x^2} \, dx \right) \left( \int_0^\infty e^{-x^2} \, dx \right) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy.
\]

Evaluate the last integral using polar coordinates and solve the resulting equation for \( I \).

**Solution.** The double integral is over the first quadrant of the Cartesian plane. So in polar coordinates we have the \( r \)-limits of 0 and \( \infty \) and the \( \theta \)-limits of 0 and \( \pi/2 \). Since \( r^2 = x^2 + y^2 \) and \( dx \, dy = r \, dr \, d\theta \) then we have: ...