Chapter 15. Multiple Integrals
15.4. Double Integrals in Polar Form—Examples and Proofs of Theorems
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Exercise 15.4.6. Describe the given region in polar coordinates:

Solution. Since \( x = r \cos \theta \) then along the vertical line \( x = 1 \) we have \( 1 = r \cos \theta \) or \( r = 1 / \cos \theta = \sec \theta \). Along the circle we have \( r = 2 \). So we can describe the region in terms of \( r \)-limits. A typical ray from the origin enters the region where \( r = \sec \theta \) and leaves where \( r = 2 \). To find \( \theta \)-limits, notice the following triangle:
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Describe the given region in polar coordinates:

Solution. Since $x = r \cos \theta$ then along the vertical line $x = 1$ we have $1 = r \cos \theta$ or $r = 1/\cos \theta = \sec \theta$. Along the circle we have $r = 2$. So we can describe the region in terms of $r$-limits. A typical ray from the origin enters the region where $r = \sec \theta$ and leaves where $r = 2$. To find $\theta$-limits, notice the following triangle:
Solution (continued). We have \( \cos \theta = 1/2 \) so that
\[ \theta = \cos^{-1}(1/2) = \pi/3. \]
So the upper \( \theta \)-limit is \( \pi/3 \) and, by symmetry, the lower \( \theta \)-limit is \( -\pi/3 \). In particular, an integral of \( f(r \theta) \) over the region is of the form
\[
\int_{-\pi/3}^{\pi/3} \int_{\sec \theta}^{2} f(r, \theta) \, dr \, d\theta.
\]
Exercise 15.4.10. Change the integral into an equivalent polar integral. Then evaluate the polar integral:

\[ \int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2) \, dx \, dy. \]

Solution. With \( x = \sqrt{1 - y^2} \) we have \( x^2 = 1 - y^2 \) where \( x \geq 0 \) and \( x^2 + y^2 = 1 \) \( x \geq 0 \). This is the upper half of the unit circle centered at the origin. With \( y \) ranging from 0 to 1 we then have the region:
Exercise 15.4.10.

Change the integral into an equivalent polar integral. Then evaluate the polar integral:

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**Solution.** With \( x = \sqrt{1-y^2} \) we have \( x^2 = 1 - y^2 \) where \( x \geq 0 \) and \( x^2 + y^2 = 1 \) \( x \geq 0 \). This is the upper half of the unit circle centered at the origin. With \( y \) ranging from 0 to 1 we then have the region:
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Solution. With \( x = \sqrt{1 - y^2} \) we have \( x^2 = 1 - y^2 \) where \( x \geq 0 \) and \( x^2 + y^2 = 1 \) \( x \geq 0 \). This is the upper half of the unit circle centered at the origin. With \( y \) ranging from 0 to 1 we then have the region:
Solution (continued). We can take the $r$-limits as 0 to 1 and the $\theta$-limits as 0 to $\pi$. Since $r^2 = x^2 + y^2$, the integral becomes

$$
\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2) \, dx \, dy = \int_0^\pi \int_0^1 r^2 r \, dr \, d\theta = \int_0^\pi \frac{1}{4} r^4 \bigg|_{r=0}^{r=1} d\theta
$$

$$
= \int_0^\pi \frac{1}{4} d\theta = \frac{1}{4} \theta \bigg|_0^\pi = \frac{\pi}{4}.
$$
Exercise 15.4.28

**Exercise 15.4.28.** Find the area of the region that lies inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 1$.

**Solution.** The graphs of the cardioid (which is Exercise #1 on page 652 in Section 11.4) and the circle are:
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Solution. The graphs of the cardioid (which is Exercise \#1 on page 652 in Section 11.4) and the circle are:

![Cardioid and Circle Graph]

So the $r$-limits of the region are 1 and $1 + \cos \theta$ and the $\theta$-limits are $-\pi/2$ and $\pi/2$. 
Exercise 15.4.28. Find the area of the region that lies inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 1$.

Solution. The graphs of the cardioid (which is Exercise #1 on page 652 in Section 11.4) and the circle are:

So the $r$-limits of the region are 1 and $1 + \cos \theta$ and the $\theta$-limits are $-\pi/2$ and $\pi/2$. 
Exercise 15.4.28 (continued)

Solution (continued). So the area of the region is

\[
A = \int_{-\pi/2}^{\pi/2} r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \int_1^{1+\cos \theta} r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \left( \frac{1}{2} r^2 \right|_{r=1}^{r=1+\cos \theta} \right) \, d\theta
\]

\[
= \int_{-\pi/2}^{\pi/2} \left( \frac{1}{2} (1 + \cos \theta)^2 - \frac{1}{2} \right) \, d\theta = \int_{-\pi/2}^{\pi/2} \left( \cos \theta + \frac{1}{2} \cos^2 \theta \right) \, d\theta
\]

\[
= \sin \theta + \frac{1}{2} \left( \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) \bigg|_{\theta=-\pi/2}^{\theta=\pi/2}
\]

by Example 9 in Section 5.5

\[
= \left( \sin(\pi/2) + \frac{1}{2} \left( \frac{(\pi/2)}{2} + \frac{\sin 2(\pi/2)}{4} \right) \right) - \left( \sin(-\pi/2) + \frac{1}{2} \left( \frac{(-\pi/2)}{2} + \frac{\sin 2(\pi/2)}{4} \right) \right)
\]

\[
= (1 + (1/2)(\pi/4 + 0)) - (-1 + (1/2)(-\pi/4 + 0)) = 2 + \pi/4.
\]
Exercise 15.4.38. Converting to a Polar Integral. Integrate
\[ f(x, y) = \frac{\ln(x^2 + y^2)}{x^2 + y^2} \]
over the region \( 1 \leq x^2 + y^2 \leq e^2 \).

Solution. We have \( r^2 = x^2 + y^2 \) so
\[ \frac{\ln(x^2 + y^2)}{x^2 + y^2} = \frac{\ln r}{r} \]
for the region is an annulus with inner radius 1 and outer radius \( e^2 \):
Exercise 15.4.38. Converting to a Polar Integral. Integrate

\[ f(x, y) = \frac{\ln(x^2 + y^2)}{x^2 + y^2} \]

over the region \(1 \leq x^2 + y^2 \leq e^2\).

Solution. We have \(r^2 = x^2 + y^2\) so \(\frac{\ln(x^2 + y^2)}{x^2 + y^2} = \frac{\ln r}{r}\). The region is an annulus with inner radius 1 and outer radius \(e^2\):
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\[ f(x, y) = \frac{\ln(x^2 + y^2)}{x^2 + y^2} \]

over the region \( 1 \leq x^2 + y^2 \leq e^2 \).

**Solution.** We have \( r^2 = x^2 + y^2 \) so 

\[ \frac{\ln(x^2 + y^2)}{x^2 + y^2} = \frac{\ln r}{r} \]

The region is an annulus with inner radius 1 and outer radius \( e^2 \):
Exercise 15.4.38 (continued)

**Solution.** We describe this with $r$-limits of 1 and $e^2$, and $\theta$-limits of 0 and $2\pi$. Since $r^2 = x^2 + y^2$ and $dx\,dy = r\,dr\,d\theta$, the integral is then:

\[
\int \int_R \frac{\ln(x^2 + y^2)}{x^2 + y^2} \,dx\,dy = \int_0^{2\pi} \int_1^{e^2} \frac{\ln r}{r} r\,dr\,d\theta = \int_0^{2\pi} \int_1^{e^2} \ln r \,dr\,d\theta
\]

\[
= \int_0^{2\pi} \left( f \ln r - r \right) \bigg|_{r=1}^{r=e^2} \,d\theta \quad \text{since} \quad \int \ln x \,dx = x \ln x - x + C
\]

by Example 2 in Section 8.1 on page 456

\[
= \int_0^{2\pi} \left( (e^2 \ln e^2 - e^2)(1 \ln 1 - 1) \right) \,d\theta = \int_0^{2\pi} \left( (2e^2 - e^2) - (0 - 1) \right) \,d\theta
\]

\[
= \int_0^{2\pi} \left( e^2 + 1 \right) \,d\theta = (e^2 + 1) \theta \bigg|_0^{2\pi} = 2\pi(e^2 + 1). \quad \square
\]
Solution. We describe this with $r$-limits of 1 and $e^2$, and $\theta$-limits of 0 and $2\pi$. Since $r^2 = x^2 + y^2$ and $dx\,dy = r\,dr\,d\theta$, the integral is then:

$$
\int_{\Omega} \frac{\ln(x^2 + y^2)}{x^2 + y^2} \, dx\,dy = \int_{0}^{2\pi} \int_{1}^{e^2} \ln r \, r\,dr\,d\theta = \int_{0}^{2\pi} \int_{1}^{e^2} \ln r \, dr\,d\theta
$$

$$
= \int_{0}^{2\pi} (f \ln r - r) \bigg|_{r=1}^{r=e^2} \, d\theta \quad \text{since} \quad \int \ln x \, dx = x \ln x - x + C
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$$
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$$
= \int_{0}^{2\pi} (e^2 + 1) \, d\theta = (e^2 + 1)\theta \bigg|_{0}^{2\pi} = 2\pi(e^2 + 1). \quad \square
$$
Exercise 15.4.41. Converting to Polar Integrals.

(a) The usual way to evaluate the improper integral \( I = \int_0^\infty e^{-x^2} \, dx \) is first to calculate its square:

\[
I^2 = \left( \int_0^\infty e^{-x^2} \, dx \right) \left( \int_0^\infty e^{-x^2} \, dx \right) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy.
\]

Evaluate the last integral using polar coordinates and solve the resulting equation for \( I \).

Solution. The double integral is over the first quadrant of the Cartesian plane. So in polar coordinates we have the \( r \)-limits of 0 and \( \infty \) and the \( \theta \)-limits of 0 and \( \pi/2 \). Since \( r^2 = x^2 + y^2 \) and \( dx \, dy = r \, dr \, d\theta \) then we have: ...
Converting to Polar Integrals.

(a) The usual way to evaluate the improper integral \( I = \int_0^\infty e^{-x^2} \, dx \) is first to calculate its square:

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Solution (continued). . .

\[ I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy = \int_0^{\pi/2} \int_0^\infty e^{-r^2} \, r \, dr \, d\theta \]

\[ = \int_0^{\pi/2} \left( \lim_{b \to \infty} \int_0^b re^{-r^2} \, dr \right) \, d\theta = \int_0^{\pi/2} \left( \lim_{b \to \infty} \left( \frac{-e^{-r^2}}{2} \right) \bigg|_{r=b} \right) d\theta \]

\[ = \int_0^{\pi/2} \lim_{b \to \infty} \left( \frac{-e^{-b^2}}{2} - \frac{-e^{-(0)^2}}{2} \right) d\theta \]

\[ = \int_0^{\pi/2} (0 + 1/2) \, d\theta = \left( \theta/2 \right) \bigg|_{\theta=0}^{\theta=\pi/2} = \pi/4. \]

So \[ I = \int_0^\infty e^{-x^2} \, dx = \sqrt{\pi/4} = \sqrt{\pi}/2. \]
Solution (continued). . .

\[ I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r \, dr \, d\theta \]

\[ = \int_0^{\pi/2} \left( \lim_{b \to \infty} \int_0^b re^{-r^2} \, dr \right) \, d\theta = \int_0^{\pi/2} \left( \lim_{b \to \infty} \left( \frac{-e^{-r^2}}{2} \right) \bigg|_{r=0}^{r=b} \right) \, d\theta \]

\[ = \int_0^{\pi/2} \lim_{b \to \infty} \left( \frac{-e^{-b^2}}{2} - \frac{-e^{-(0)^2}}{2} \right) \, d\theta \]

\[ = \int_0^{\pi/2} (0 + 1/2) \, d\theta = \left( \frac{\theta}{2} \right) \bigg|_{\theta=0}^{\theta=\pi/2} = \frac{\pi}{4}. \]

So \( I = \int_0^\infty e^{-x^2} \, dx = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}. \)
Exercise 15.4.41. Converting to Polar Integrals.

(b) Evaluate

$$\lim_{x \to \infty} e^{f(x)} = \lim_{x \to \infty} \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}} \, dt.$$ 

Solution. We have

$$\lim_{x \to \infty} e^{f(x)} = \lim_{x \to \infty} \left( \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}} \, dt \right) = \frac{2}{\sqrt{\pi}} \left( \lim_{x \to \infty} \int_0^x e^{-t^2} \, dt \right)$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} \, dt = \frac{2}{\sqrt{\pi}} \left( \frac{\sqrt{\pi}}{2} \right) \text{ by part (a)}$$

$$= 1. \quad \square$$
Exercise 15.4.41. Converting to Polar Integrals.

(b) Evaluate

\[ \lim_{x \to \infty} e^{-f(x)} = \lim_{x \to \infty} \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}} \, dt. \]

Solution. We have

\[
\lim_{x \to \infty} \text{erf}(x) = \lim_{x \to \infty} \left( \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}} \, dt \right) = \frac{2}{\sqrt{\pi}} \left( \lim_{x \to \infty} \int_0^x e^{-t^2} \, dt \right)
\]

\[ = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} \, dt = \frac{2}{\sqrt{\pi}} \left( \frac{\sqrt{\pi}}{2} \right) \text{ by part (a)} \]

\[ = 1. \]