Chapter 11. Parametric Equations and Polar Coordinates

11.4. Graphing in Polar Coordinates

Note. The text gives the following as "Symmetry Tests for Polar Graphs:"

1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph.

2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph.

3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph.

This is, however, misleading. First, notice that $(r, -\theta)$ and $(-r, \pi - \theta)$ are representations of the same point (as are $(r, \pi - \theta)$ and $(-r, -\theta)$, and $(-r, \theta)$ and $(r, \theta + \pi)$)—one with $r > 0$ and one with $r < 0$). However, we cannot use these representations to *algebraically* test for symmetries. We can use these representations to check for symmetries once we have a graph, but the purpose of studying symmetries is to *assist* in graphing. It may be, for example, that the point $(r, -\theta)$ lies on the graph whenever

(r, θ) does (meaning symmetry with respect to the x-axis), but that *some other representation* of point $(r, -\theta)$ satisfies the equation determining the graph. Any representation of point $(r, -\theta)$ is either of the form $(r, -\theta + 2n\pi)$ or of the form $(-r, \pi - \theta + 2n\pi)$ where n is some integer ($n \in \mathbb{Z}$). Therefore, to actually *test an equation* for symmetry (as the instructions to the homework problems require), we need the following, which take into consideration all representations of the points which the text mentions:

Symmetry Tests for Polar Graphs of $r = f(\theta)$.

1. *Symmetry about the x-axis:* If the point (r, θ) satisfies the relationship $r = f(\theta)$, then some point of the form $(r, -\theta + 2n\pi)$ or $(-r, \pi - \theta + 2n\pi)$ satisfies the relationship for some $n \in \mathbb{Z}$. That is, $r = f(-\theta + 2n\pi)$ for some $n \in \mathbb{Z}$ or $-r = f(\pi - \theta + 2n\pi)$ for some $n \in \mathbb{Z}$

2. *Symmetry about the y-axis:* If the point (r, θ) satisfies the relationship $r = f(\theta)$, then some point of the form $(r, \pi - \theta + 2n\pi)$ or $(-r, -\theta + 2n\pi)$ satisfies the relationship for some $n \in \mathbb{Z}$. That is, $r = f(\pi - \theta + 2n\pi)$ for some $n \in \mathbb{Z}$ or $-r = f(-\theta + 2n\pi)$ for some $n \in \mathbb{Z}$.

3. *Symmetry about the origin:* If the point (r, θ) satisfies the relationship $r = f(\theta)$, then some point of the form $(-r, \theta + 2n\pi)$ or $(r, \theta + \pi + 2n\pi)$ satisfies the relationship for some $n \in \mathbb{Z}$. That is, $-r = f(\theta + 2n\pi)$

for some $n \in \mathbb{Z}$ or $r = f(\theta + \pi + 2n\pi)$ for some $n \in \mathbb{Z}$.

We illustrate this with an example, and show the necessity of using our symmetry test.

Example. Page 652, number 8. Test $r = \cos(\theta/2)$ for symmetries with respect to the x- and y-axes.

Solution. (1) To test for symmetry with respect to the x-axis, suppose the point (r, θ) lies on the graph of $r = f(\theta)$. We first test to see if $(r, -\theta + 2n\pi)$ lies on the graph for some $n \in \mathbb{Z}$. We replace "r with r" and replace "θ with $-\theta + 2n\pi$" in $r = f(\theta)$ to see if we get an equation consistent with $r = f(\theta)$ for some $n \in \mathbb{Z}$:

$$\begin{aligned} r &= \cos(\theta/2) \\ r &= \cos((-\theta + 2n\pi)/2)? \text{ (replacing } r \text{ and } \theta \text{ as needed)} \\ &= \cos((-\theta/2) + n\pi)? \\ &= \cos(-\theta/2)\cos(n\pi) - \sin(-\theta/2)\sin(n\pi)? \\ &= \cos(\theta/2)\cos(n\pi) - \sin(\theta/2) \cdot 0? \\ &= \cos(\theta/2)\cos(n\pi)? \end{aligned}$$

This reduces to the original equation $r = \cos(\theta/2)$ if $\cos(n\pi) = 1$. This is the case if $n = 0, \pm 2, \pm 4, \pm 6, \ldots$. Therefore, this equation does have symmetry with respect to the x-axis. **Two Observations:** (a) We have established the symmetry by considering $(r, -\theta + 2n\pi)$, so there is no need to consider the representation $(-r, \pi - \theta + 2n\pi)$, and (b) Since we can use $n = 0$ to establish the symmetry above, then this means that the point $(r, -\theta)$ lies on the graph and *the text's test for symmetry would have worked here*!

(2) To test for symmetry with respect to the y-axis, suppose the point (r, θ) lies on the graph of $r = f(\theta)$. We first test to see if $(r, \pi - \theta + 2n\pi)$ lies on the graph for some $n \in \mathbb{Z}$. We replace "r with r" and replace "θ with $\pi - \theta + 2n\pi$" in $r = f(\theta)$ to see if we get an equation consistent with $r = f(\theta)$ for some $n \in \mathbb{Z}$:

$$\begin{aligned}
r &= \cos(\theta/2) \\
r &= \cos((\pi - \theta + 2n\pi)/2)? \text{ (replacing } r \text{ and } \theta \text{ as needed)} \\
&= \cos((\pi/2 - \theta/2) + n\pi)? \\
&= \cos(\pi/2 - \theta/2)\cos(n\pi) - \sin(\pi - \theta/2)\sin(n\pi)? \\
&= \sin(\theta/2)\cos(n\pi) - \cos(\theta/2) \cdot 0? \\
&= \sin(\theta/2)\cos(n\pi)?
\end{aligned}$$

Since $\cos(n\pi) \in \{-1, 0, 1\}$, then we cannot make this consistent with the original equation. Since this test has failed, we must also test the point $(-r, -\theta + 2n\pi)$. We replace "r with $-r$" and replace "θ with $-\theta + 2n\pi$" in $r = f(\theta)$ to see if we get an equation consistent with $r = f(\theta)$ for some $n \in \mathbb{Z}$:

$$\begin{aligned} r &= \cos(\theta/2) \\ -r &= \cos((-\theta + 2n\pi)/2)? \text{ (replacing } r \text{ and } \theta \text{ as needed)} \\ &= \cos((-\theta/2) + n\pi)? \\ &= \cos(-\theta/2)\cos(n\pi) - \sin(-\theta/2)\sin(n\pi)? \\ &= \cos(\theta/2)\cos(n\pi) + \cos(\theta/2) \cdot 0? \\ &= \cos(\theta/2)\cos(n\pi)? \end{aligned}$$

This reduces to the original equation $r = \cos(\theta/2)$ if $\cos(n\pi) = -1$. This is the case if $n = \pm 1, \pm 3, \pm 5, \ldots$. Therefore, this equation does have symmetry with respect to the y-axis. **Two Observations:** (a) We had to consider *both representations* of points to see that there actually is symmetry with respect to the y-axis, and (b) Since we must use $n = \pm 1, \pm 3, \pm 5, \ldots$ in the point $(-r, -\theta + 2n\pi)$ to establish the symmetry, we cannot use the representation which the text's test gives, $(-r, -\theta)$ (where we would need $n = 0$), and *the text's test for symmetry would fail in the formula* $r = f(\theta)$!

Note. With $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$, then by the Chain Rule:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{d[f(\theta)\cos\theta]/d\theta}{d[f(\theta)\sin\theta]/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

assuming $dx/d\theta \neq 0$.

Example. Page 652, number 20.

Note. One technique for graphing in polar coordinates is to (1) graph $r = f(\theta)$ in the Cartesian (r,θ)-plane, and then (2) use the Cartesian graph as a guide to sketch the polar coordinate graph.

Note. Graphs of $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ determine *limaçons* (the French word for snail). There are four basic shapes for limaçons: (1) limaçon with an inner loop, (2) cardioids, (3) dimpled lamaçons, and (4) oval limaçons.

Example. Page 653, number 22a.