Chapter 11. Parametric Equations and Polar Coordinates

11.5. Areas and Lengths in Polar Coordinates

Note. Recall that a sector of a circle with radius r and central angle θ has area $\frac{1}{2}r^2\theta$. The region OTS in Figure 11-30 below is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. We partition the interval $[\alpha, \beta]$ into n pieces. The typical sector (a "$d\theta$-slice") has radius $r_k = f(\theta_k)$ and central angle of measure $\Delta\theta_k$. So this sector has area $A_k = \dfrac{1}{2}r_k^2\Delta\theta_k = \dfrac{1}{2}(f(\theta_k))^2\Delta\theta_k$. Summing up the areas of the sectors produces a Riemann sum. Letting the limit of the norm of the partition approach 0 yields the exact area:

$$A = \int_\alpha^\beta \frac{1}{2}r^2\,d\theta = \int_\alpha^\beta \frac{1}{2}(f(\theta))^2\,d\theta.$$

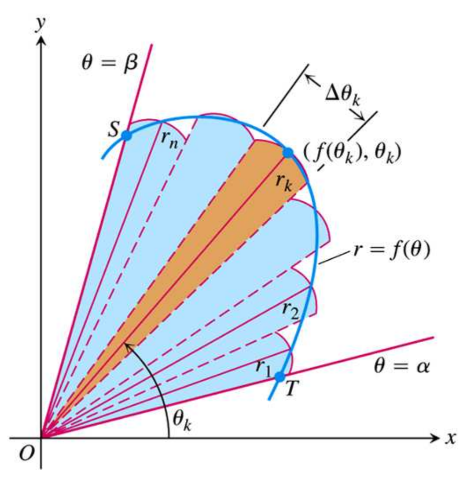

Figure 11.30, page 653

Example. Page 656, number 6.

Note. If $r = f(\theta)$, then we can treat θ as the parameter, and say that the rectangular coordinates x and y are determined parametrically as $x = r\cos\theta = f(\theta)\cos\theta$ and $y = r\sin\theta = f(\theta)\sin\theta$. The length of $r = f(\theta)$ for $\theta \in [\alpha, \beta]$ (where the curve determined by $r = f(\theta)$ is traced out exactly once for $\theta \in [\alpha, \beta]$) is:

$$\begin{aligned}
L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta \\
&= \int_\alpha^\beta \sqrt{(f'(\theta)\cos\theta + f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta - f(\theta)\cos\theta)^2}\,d\theta \\
&= \int_\alpha^\beta \sqrt{(f'(\theta))^2\cos^2\theta + 2f'(\theta)f(\theta)\cos\theta\sin\theta + (f(\theta))^2\sin^2\theta} \\
&\qquad \overline{+(f'(\theta))^2\sin^2\theta - 2f'(\theta)f(\theta)\sin\theta\cos\theta + (f(\theta)^2\cos^2\theta}\,d\theta \\
&= \int_\alpha^\beta \sqrt{(f'(\theta))^2 + (f(\theta))^2}\,d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta.
\end{aligned}$$

Example. Page 657, number 22.