Chapter 11. Parametric Equations and Polar Coordinates

11.5. Areas and Lengths in Polar Coordinates

Note. Recall that a sector of a circle with radius $r$ and central angle $\theta$ has area $\frac{1}{2}r^2\theta$. The region $OTS$ in Figure 11-30 below is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. We partition the interval $[\alpha, \beta]$ into $n$ pieces. The typical sector (a “$d\theta$-slice”) has radius $r_k = f(\theta_k)$ and central angle of measure $\Delta \theta_k$. So this sector has area $A_k = \frac{1}{2} r_k^2 \Delta \theta_k = \frac{1}{2} (f(\theta_k))^2 \Delta \theta_k$. Summing up the areas of the sectors produces a Riemann sum. Letting the limit of the norm of the partition approach 0 yields the exact area:

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} (f(\theta))^2 \, d\theta.$$
Example. Page 656, number 6.

Note. If \( r = f(\theta) \), then we can treat \( \theta \) as the parameter, and say that the rectangular coordinates \( x \) and \( y \) are determined parametrically as \( x = r \cos \theta = f(\theta) \cos \theta \) and \( y = r \sin \theta = f(\theta) \sin \theta \). The length of \( r = f(\theta) \) for \( \theta \in [\alpha, \beta] \) (where the curve determined by \( r = f(\theta) \) is traced out exactly once for \( \theta \in [\alpha, \beta] \)) is:

\[
L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta
\]

\[
= \int_{\alpha}^{\beta} \sqrt{(f'(\theta) \cos \theta + f(\theta) \sin \theta)^2 + (f'(\theta) \sin \theta - f(\theta) \cos \theta)^2} \, d\theta
\]

\[
= \int_{\alpha}^{\beta} \sqrt{(f'(\theta))^2 \cos^2 \theta + 2f'(\theta)f(\theta) \cos \theta \sin \theta + (f(\theta))^2 \sin^2 \theta}
\]

\[
+ (f'(\theta))^2 \sin^2 \theta - 2f'(\theta)f(\theta) \sin \theta \cos \theta + (f(\theta))^2 \cos^2 \theta \, d\theta
\]

\[
= \int_{\alpha}^{\beta} \sqrt{(f'(\theta))^2 + (f(\theta))^2} \, d\theta = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta.
\]

Example. Page 657, number 22.