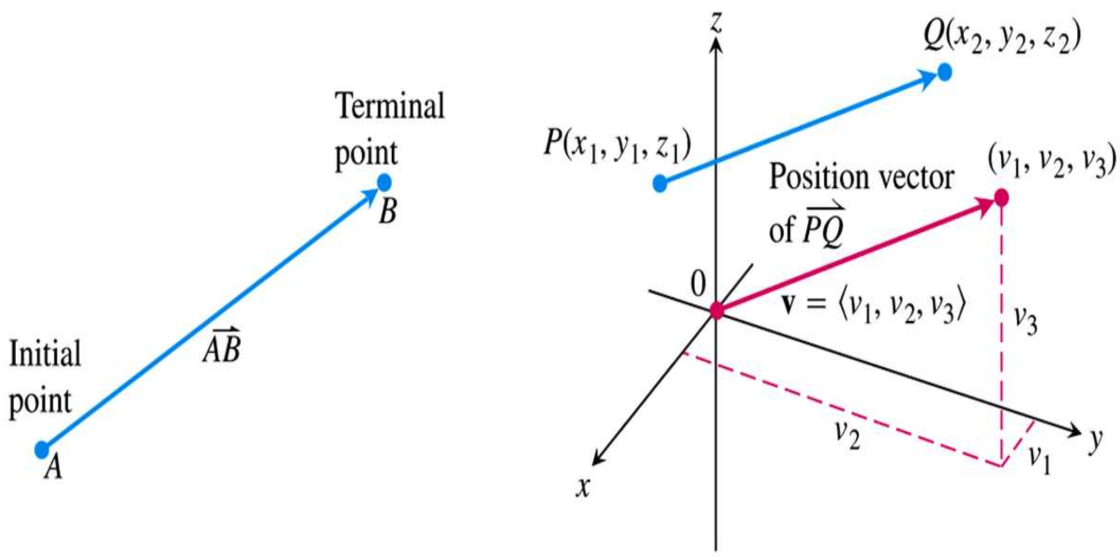
Chapter 12. Vectors and the Geometry of Space

12.2. Vectors

Note. Several physical quantities are represented by an entity which involves both magnitude and direction. Examples of such entities are force, velocity, acceleration, torque, and angular momentum (and sometimes position). In here (i.e., Calculus 3), we use these applications to motivate our definitions. In a Linear Algebra class (MATH 2010—see `http://faculty.etsu.edu/gardnerr/2010/notes.htm` for a set of notes for the Linear Algebra class), you will take a more formal approach and a vector will be something more general than it is here.

Definition. The *vector* from point A to point B is the directed line segment from A to B and is denoted $\vec{AB}$. Point A is the *initial point* and point B is the *terminal point* of vector $\vec{AB}$.



Figures 12.7 and 12.10, page 683

Note. Though not yet defined, a vector will only have *magnitude* and *direction*. It will not have a *position*! Geometrically, think of a vector as an arrow which can be translated around, but which can be neither stretched nor rotated. If we translate a vector so that its initial point is at the origin of a Cartesian coordinate system, then the vector is said to be in *standard position* (see Figure 12.10 above).

Definition. When a vector is in standard position, it will then have as its terminal point, some point (v_1, v_2, v_3) (or some point (v_1, v_2) if the vector is in two-dimensions). The *component form* of this vector is then $\langle v_1, v_2, v_3 \rangle$ (or $\langle v_1, v_2 \rangle$ if the vector is in two-dimensions). The numbers v_1, v_2, and v_3 are the *components* of $\mathbf{v}$. *In these notes* (and in the text), we will use bold-faced fonts to represent vectors. For example, we represent vector $\langle v_1, v_2, v_3 \rangle$ as $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. *On the whiteboard* we use a little arrow over the letter which represents the vector: $\vec{v} = \langle v_1, v_2, v_3 \rangle$.

Note. It now follows that the vector from point $P(x_1, y_1, z_1)$ to point $Q(x_2, y_2, z_2)$ is $\mathbf{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$.

Note. Notice that there is a vital difference between a vector and a point!!! A vector has a magnitude and direction, but no position! A point has a position, but neither magnitude nor direction! Hence, we **must** have a notation which distinguishes between the two. That is why we use parentheses to represent points (the *point* (x, y, z)) and angled brackets to represent vectors (the *vector* $\langle x, y, z \rangle$).

Definition. Two vectors are *equal* if they have the same component form. The *magnitude* (or *length*) of vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Notice that this is the distance between endpoints of vector $\mathbf{v}$.

Example. Page 690, number 26.

Note. We now explore the *algebraic* properties of vectors. You will see this again (and more formally) if you take our Linear Algebra class (MATH 2010). To do so, we note that there are two fundamentally different objects which we consider in this section: vectors and numbers (which, in this context, will be called *scalars*). We will add vectors together and multiply vectors by scalars.

Definition. Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with k a scalar (i.e., number). Then define:

Vector Addition: $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$,

Scalar Multiplication: $k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$.

Of course, similar definitions hold when the vector is two-dimensional.

Note. The definition of vector addition can be illustrated *geometrically* in terms of the following diagram (Figure 12.12). These diagrams illustrate the fact that vectors follow a *parallelogram law* of addition. The vector $\mathbf{u} + \mathbf{v}$ is called the *resultant vector* of the vector addition. We define the *difference* $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1\mathbf{v})$.

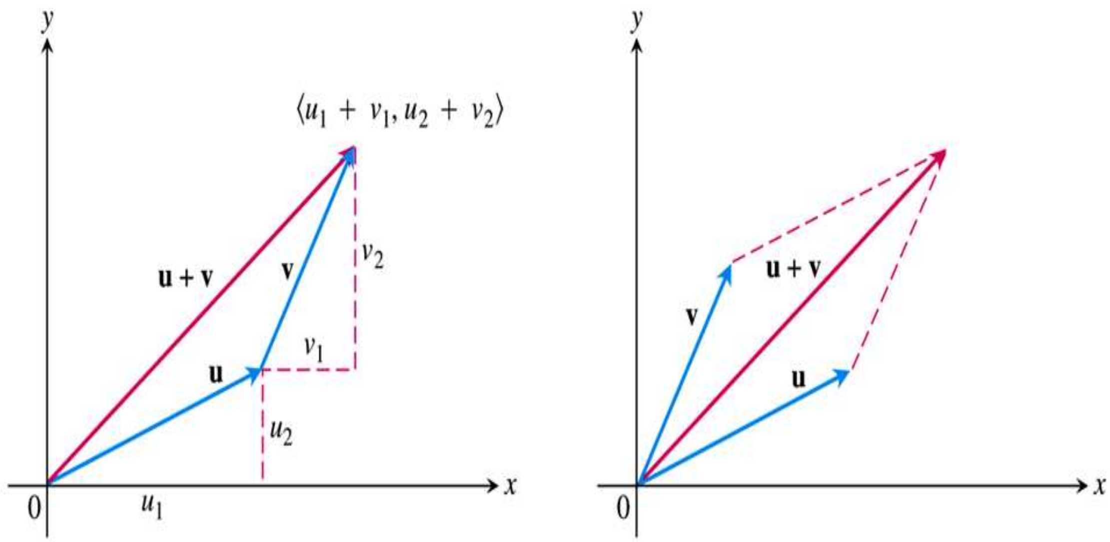


Figure 12.12, page 685

Note. Scalar addition can be illustrated *geometrically* in terms of the following diagram (Figure 12.13). Notice that the scalar stretches (or shrinks) the magnitude of the original vector and if the scalar is negative, then it reverses the direction of the original vector.

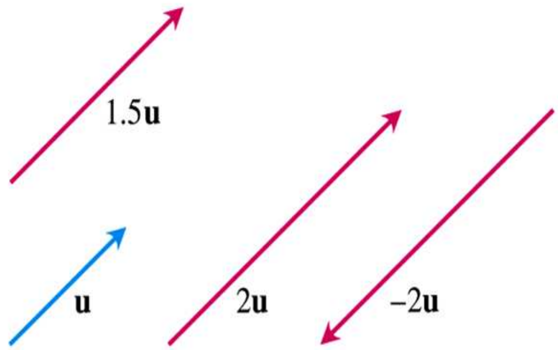


Figure 12.13, page 686

Example. Page 690, number 24.

Theorem (Properties of Vector Operations). Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors and a, b be scalars. Then

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ (Commutative Property of Vector Addition).

2. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ (Associative Property of Vector Addition).

3. $\mathbf{u}+\mathbf{0}=\mathbf{u}$ (The Zero Vector $\mathbf{0}$ is the Additive Identity under Vector Addition).

4. $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$ (The Additive Inverse of Vector $\mathbf{u}$ is $-\mathbf{u}$ under Vector Addition).

5. $0\mathbf{u} = \mathbf{0}$ (Behavior of Scalar 0 in Scalar Multiplication).

6. $1\mathbf{u} = \mathbf{u}$ (Behavior of Scalar 1 in Scalar Multiplication).

7. $a(b\mathbf{u}) = (ab)\mathbf{u}$ (Associativity of Scalar Multiplication).

8. $a(\mathbf{u}+\mathbf{v}) = a\mathbf{u}+a\mathbf{v}$ (Distribution of Scalar Multiplication over Vector Addition).

9. $(a+b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$ (Distribution of Scalar Addition over Scalar Multiplication).

Example. Page 690, number 22.

Definition. A vector $\mathbf{v}$ of length 1 is called a *unit vector*. The three *standard unit vectors* are:

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

Note. Any vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ can be written as a *linear combination* of the standard unit vectors as

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

The use of the word *linear* here is the same as its use in the class titled "*Linear* Algebra." We call v_1 the $\mathbf{i}$-*component*, v_2 the $\mathbf{j}$-*component*, and v_3 the **k**-*component*. In component form, the vector from point $P_1(x_1, y_1, z_1)$ to point $P_2(x_2, y_2, z_2)$ is

$$\vec{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$

In standard position, this vector has its tail at the origin and its head at the point $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

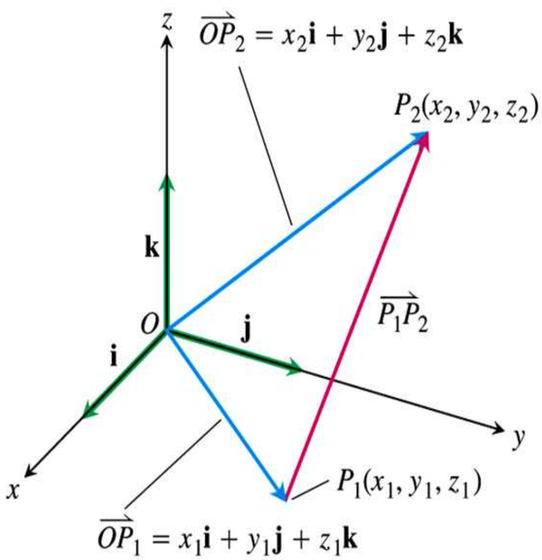


Figure 12.15, page 687

Definition. The *direction* of nonzero vector $\mathbf{v}$ is the unit vector $\mathbf{v}/|\mathbf{v}|$.

Example. Page 691, number 36a.

Definition. The *midpoint* M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

Example. Page 691, number 36b.

Example. Page 691, number 46 (an application similar to those seen in physics).