Chapter 12. Vectors and the Geometry of Space

12.3. The Dot Product

Note. In this section we introduce an operation which can be performed on two vectors. The operation is called *dot product*, or sometimes *inner product* or *scalar product*. We use this product to measure angles between vectors.

Theorem 1. Angle Between Two Vectors. The angle θ between two nonzero vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}|\,|\mathbf{v}|}\right).$$

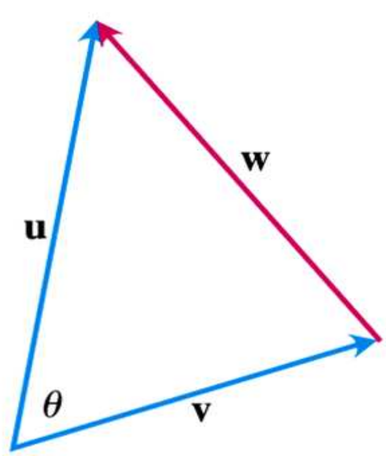

Figure 12.21, page 693

Proof. Referring to Figure 12.21 above, we have by the Law of Cosines that

$$|\mathbf{w}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\,|\mathbf{v}|\cos\theta, \text{ or}$$

$$2|\mathbf{u}|\,|\mathbf{v}|\cos\theta = |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2.$$

In terms of components, $\mathbf{w} = \mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3\rangle$. So

$$\begin{aligned}
|\mathbf{u}|^2 &= u_1^2 + u_2^2 + u_3^2 \\
|\mathbf{v}|^2 &= v_1^2 + v_2^2 + v_3^2 \\
|\mathbf{w}|^2 &= (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2 \\
&= u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2 + u_3^2 - 2u_3v_3 + v_3^2.
\end{aligned}$$

These equations combine to give

$$|\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = 2(u_1v_1 + u_2v_2 + u_3v_3).$$

Therefore,

$$2|\mathbf{u}|\,|\mathbf{v}|\cos\theta = |\mathbf{u}|^2 + |\mathbf{v}|^2 - |\mathbf{w}|^2 = (2(u_1v_1 + u_2v_2 + u_3v_3) \text{ and}$$

$$\cos\theta = \frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}|\,|\mathbf{v}|}.$$

Since $\theta \in [0, \pi)$, we have

$$\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}|\,|\mathbf{v}|}\right).$$

Q.E.D.

Definition. The *dot product* of two vectors $\mathbf{u} = \langle u_1, u_2, v_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

If the vectors only have two components, then the dot product is similarly defined.

Note. Notice that the dot product of two vectors is *not* a vector, but a scalar (and that's why the dot product is sometimes called a "scalar product"). In terms of dot products, the angle θ between vectors $\mathbf{u}$ and $\mathbf{v}$ are

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right).$$

Example. Page 698, number 14.

Note. We will be particularly interested in the situation when vectors are perpendicular. That is, when the angle between the vectors is $\pi/2$. Since $\cos(\pi/2) = 0$, we have the following definition.

Definition. Vectors $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* (or *perpendicular*) if an only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Theorem. Properties of the Dot Product. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and c is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ (Commutative Property of Dot Product).

2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$ (Distribution of scalar Multiplication through Dot Product).

3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (Distribution of Dot Product over Vector Addition).

4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$.

5. $\mathbf{0} \cdot \mathbf{u} = 0$.

Each of these properties is easily verified by computations with the components of the vectors.

Note. In applications (and theoretical problems), it is often desired to find the "piece" of a vector which goes in a certain direction. That is, we desire to find the *projection* of one vector $\mathbf{u}$ onto another $\mathbf{v}$ (where the projection is denoted $\text{proj}_{\mathbf{v}}(\mathbf{u})$), as illustrated in Figure 12.25 below.

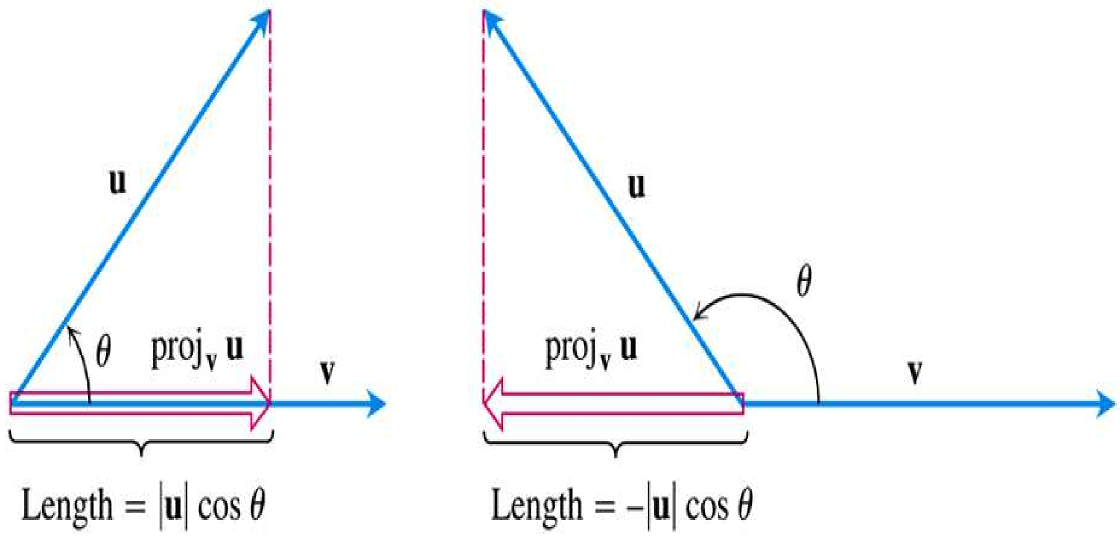

Figure 12.25, page 696

Definition. The vector projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}.$$

Note. Notice that the projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector with scalar component $\dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = |\mathbf{u}|\cos\theta$ (see Figure 12.25) and direction $\mathbf{v}/|\mathbf{v}|$.

Example. Page 699, number 24.

Note. Recall from physics, that *work* equals *force* times *distance*. More formally, work is the projection of the force vector onto the line determining the displacement. Since projections are computed using dot products, then so is work.

Definition. The *work* done by a constant force $\mathbf{F}$ acting through a displacement $\mathbf{D} = \vec{PQ}$ is $W = \mathbf{F} \cdot \mathbf{d}$.

Example. Page 699, number 43.