Chapter 12. Vectors and the Geometry of Space

12.5. Lines and Planes in Space

Note. In the plane, a line is determined by a point and a number giving the slope of the line. In space, a line is determined by a point and a *vector* giving the direction of the line.

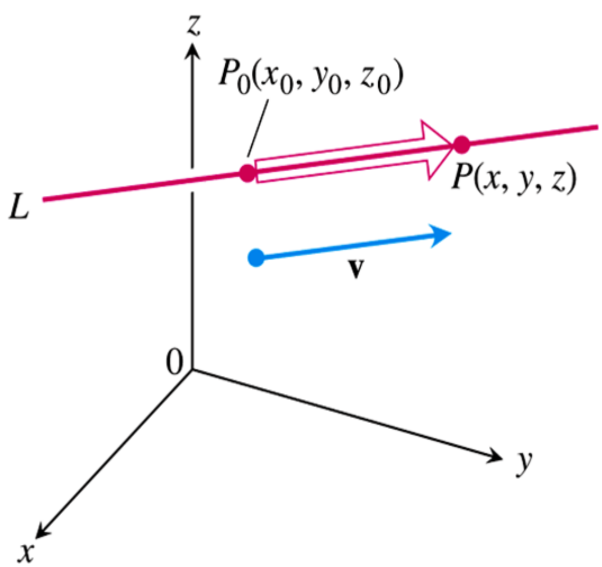

Figure 12.35, page 706

Definition. The *vector equation for the line* L through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v}$ is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \;\; t \in (-\infty, \infty),$$

where $\mathbf{r}$ is the position vector of a point $P(x, y, z)$ on L and $\mathbf{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$ (and so $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$).

Definition. The *standard parametrization of the line* through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$x = x_0 + tv_1, \; y = y_0 + tv_2, \; z = z_0 + tv_3, \; t \in (-\infty, \infty).$$

Example. Page 712, number 16.

Note. To find the distance from a point S to a line that passes through a point P parallel to a vector $\mathbf{v}$, we find the absolute value of the scalar component of $\vec{PS}$ in the direction of a vector normal to the line. As given in Figure 12.38 below, this value is $|\vec{PS}|\sin\theta = \dfrac{|\vec{PS} \times \mathbf{v}|}{|\mathbf{v}|}$.

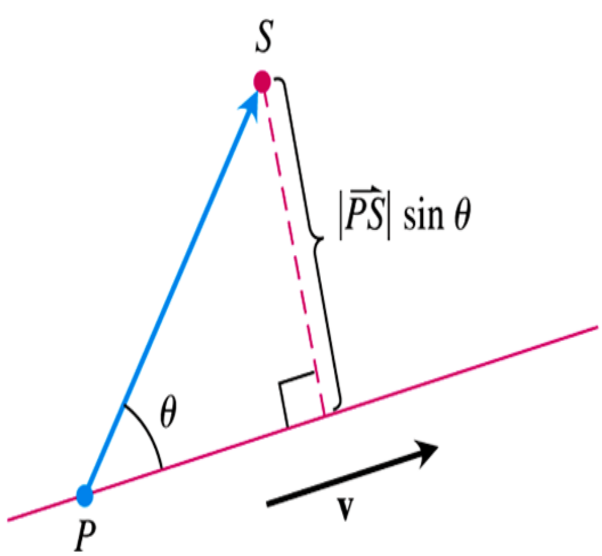

Figure 12.38, page 708

Example. Page 713, number 36.

Note. Suppose that plane M passes through a point $P_0(x_0, y_0, z_0)$ and is normal to the nonzero vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$. Then M is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$. Thus, the dot product $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$. This yields

$$(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot ((x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k}) = 0$$

or

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

So the equation of a plane is determined by a point P_0 and a normal vector $\mathbf{n}$.

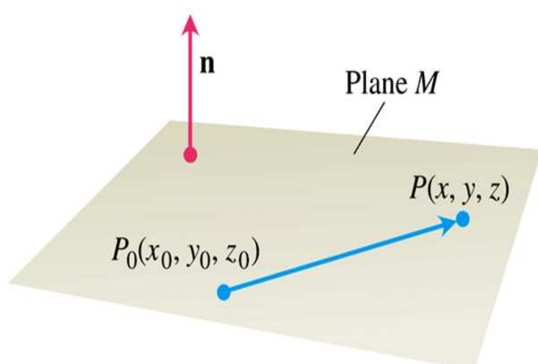

Figure 12.39, page 709

Example. Page 713, number 22.

Example. Page 713, number 58.

Note. If P is a point on a plane with normal vector $\mathbf{n}$, then the distance from any point S to the plane is the length of the vector projection of $\vec{PS}$ onto $\mathbf{n}$. This distance is $|\text{proj}_{\mathbf{n}}\vec{PS}| = \left|\vec{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right|$.

Example. Page 713, number 40.

Example. Page 714, number 70.