Chapter 13. Vector-Valued Functions and Motion in Space

13.1. Curves in Space and Their Tangents

Note. When a particle moves through space during a time interval I, we think of the particle's coordinates as functions defined on I:

$$x = f(t),\ y = g(t),\ z = h(t),\ t \in I.$$

The points $(x, y, z) = (f(t), g(t), h(t))$, $t \in I$, make up the *curve* in space that we call the particle's *path*. The above equations *parametrize* the curve. A curve in space can also be represented in vector form. The vector

$$\mathbf{r}(t) = \vec{OP} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

from the origin to the particle's *position* $P(f(t), g(t), h(t))$ at time t.

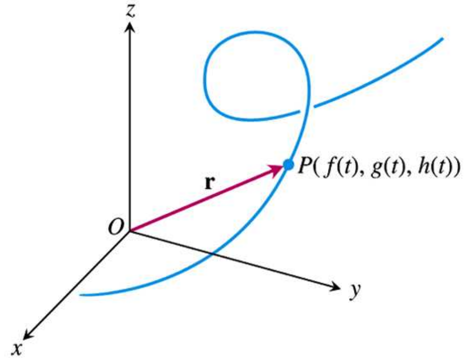

Figure 13.1, page 725

Example. Page 726, Example 1. Consider the vector function $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$. This curve is a helix.

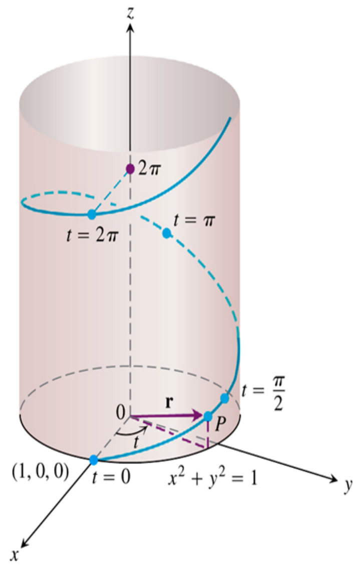

Figure 13.3, page 726

Definition. Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function defined on an open interval containing t_0 except possibly at t_0 itself, and let $\mathbf{L}$ a vector. We say that $\mathbf{r}$ has *limit* $\mathbf{L}$ as t approaches t_0 and write $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$ if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$|\mathbf{r}(t) - \mathbf{L}| < \epsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta.$$

Definition. A vector function $\mathbf{r}(t)$ is *continuous at a point* $t = t_0$ in its domain if $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function is *continuous* if it is continuous at every point in its domain.

Definition. The vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ has a *derivative* at t if f, g, and h have derivatives at t. The derivative is the vector function

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{i} + \frac{dh}{dt}\mathbf{k}.$$

The curve traced by $\mathbf{r}$ is *smooth* if $d\mathbf{r}/dt$ is continuous and never $\mathbf{0}$, that is, if f, g, and h have continuous first derivatives that are not simultaneously 0.

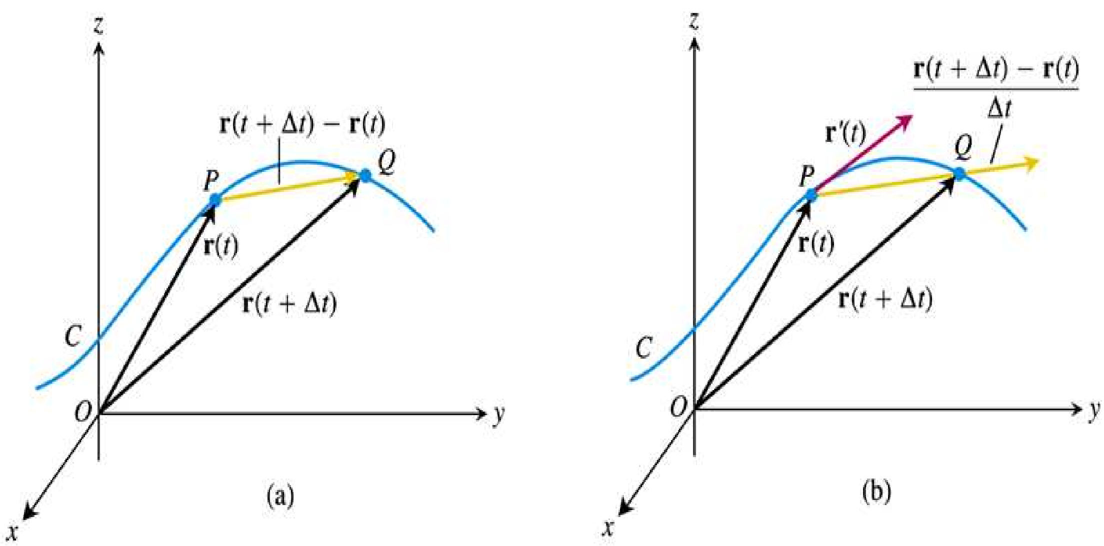

Figure 13.5, page 728

Definition. The vector $\mathbf{r}'(t)$, when different from $\mathbf{0}$, is defined to be the vector *tangent* to the curve at P. The *tangent line* to the curve at a point $(f(t_0), g(t_0), h(t_0))$ is defined to the the line through the point parallel to $\mathbf{r}'(t_0)$.

Example. Page 732, number 22.

Definition. If $\mathbf{r}$ is the position vector of a particle moving along a smooth curve in space, then $\mathbf{v}(t) = \dfrac{d\mathbf{r}}{dt}$ is the particle's *velocity vector*, tangent to the curve. At any time, the direction of $\mathbf{v}$ is the *direction of motion*, the magnitude of $\mathbf{v}$ is the particle's *speed*, and the derivative $\mathbf{a} = d\mathbf{v}/dt$, when it exists, is the particles *acceleration vector*. In summary,

1. Velocity is the derivative of position: $\mathbf{v} = \dfrac{d\mathbf{r}}{dt}$.

2. Speed is the magnitude of velocity: Speed $= |\mathbf{v}|$.

3. Acceleration is the derivative of velocity: $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2}$.

4. The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of motion at time t.

Example. Page 732, number 8.

Theorem. Differentiation Rules for Vector Functions.

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector functions of t, $\mathbf{C}$ a constant vector, c any scalar, and f any differentiable scalar function.

1. *Constant Function Rule:* $\dfrac{d}{dt}[\mathbf{C}] = \mathbf{0}$.

2. *Scalar Multiple Rules:* $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$.

$$\frac{d}{dt}[f(t)\mathbf{u}(t)] = [f'(t)](\mathbf{u}(t)) + (f(t))[\mathbf{u}'(t)].$$

3. *Sum Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$.

4. *Difference Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$.

5. *Dot Product Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = [\mathbf{u}'(t)] \cdot (\mathbf{v}(t)) + (\mathbf{u}(t)) \cdot [\mathbf{v}'(t)]$.

6. *Cross Product Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = [\mathbf{u}'(t)] \times (\mathbf{v}(t)) + (\mathbf{u}(t)) \times [\mathbf{v}'(t)]$.

7. *Chain Rule:* $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$.

Proof of the Dot Product Rule.

Suppose that $\mathbf{u} = u_1(t)\mathbf{i} + u_2(t)\mathbf{j} + u_3(t)\mathbf{k}$ and $\mathbf{v} = v_1(t)\mathbf{i} + v_2(t)\mathbf{j} + v_3(t)\mathbf{k}$.

Then

$$\begin{aligned}
\frac{d}{dt}[\mathbf{u} \cdot \mathbf{v}] &= \frac{d}{dt}[u_1v_1 + u_2v_2 + u_3v_3] \\
&= [u_1'](v_1) + (u_1)[v_1'] + [u_2'](v_2) + (u_2)[v_2'] + [u_3'](v_3) + (u_3)[v_3'] \\
&= [u_1'](v_1) + [u_2'](v_2) + [u_3'](v_3) + (u_1)[v_1'] + (u_2)[v_2'] + (u_3)[v_3'] \\
&= \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}'.
\end{aligned}$$

Proof of the Cross Product Rule.

This proof resembles the Product Rule from Calculus 1. By definition,

$$\frac{d}{dt}[\mathbf{u} \times \mathbf{v}] = \lim_{h \to 0} \frac{\mathbf{u}(t+h) \times \mathbf{v}(t+h) - \mathbf{u}(t) \times \mathbf{v}(t)}{h}.$$

This leads to

$$\frac{d}{dt}[\mathbf{u} \times \mathbf{v}] = \lim_{h \to 0} \frac{\mathbf{u}(t+h) \times \mathbf{v}(t+h) - \mathbf{u}(t) \times \mathbf{v}(t)}{h}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{\mathbf{u}(t+h)\times\mathbf{v}(t+h) - \mathbf{u}(t)\times\mathbf{v}(t+h) + \mathbf{u}(t)\times\mathbf{v}(t+h) - \mathbf{u}(t)\times\mathbf{v}(t)}{h} \\
&= \lim_{h\to 0} \left[\frac{\mathbf{u}(t+h)-\mathbf{u}(t)}{h}\times\mathbf{v}(t+h) + \mathbf{u}(t)\times\frac{\mathbf{v}(t+h)-\mathbf{v}(t)}{h}\right] \\
&= \lim_{h\to 0} \frac{\mathbf{u}(t+h)-\mathbf{u}(t)}{h} \times \lim_{h\to 0}\mathbf{v}(t+h) + \lim_{h\to 0}\mathbf{u}(t)\times\lim_{h\to 0}\frac{\mathbf{v}(t+h)-\mathbf{v}(t)}{h} \\
&= [\mathbf{u}'(t)]\times(\mathbf{v}(t)) + (\mathbf{u}(t))\times[\mathbf{v}'(t)].
\end{aligned}
$$

We have used the fact that the limit of a product is the product of the limits (Exercise 32) and that $\mathbf{v}$ is continuous and hence $\lim_{h\to 0}\mathbf{v}(t+h)=\mathbf{v}(t)$.

Example. Page 732, numbers 28a.