Chapter 13. Vector-Valued Functions and Motion in Space

13.6. Velocity and Acceleration in Polar Coordinates

Definition. When a particle $P(r, \theta)$ moves along a curve in the polar coordinate plane, we express its position, velocity, and acceleration in terms of the moving unit vectors

$$\mathbf{u}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}, \qquad \mathbf{u}_\theta = -(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}.$$

The vector $\mathbf{u}_r$ points along the position vector $\vec{OP}$, so $\mathbf{r} = r\mathbf{u}_r$. The vector $\mathbf{u}_\theta$, orthogonal to $\mathbf{u}_r$, points in the direction of increasing θ.

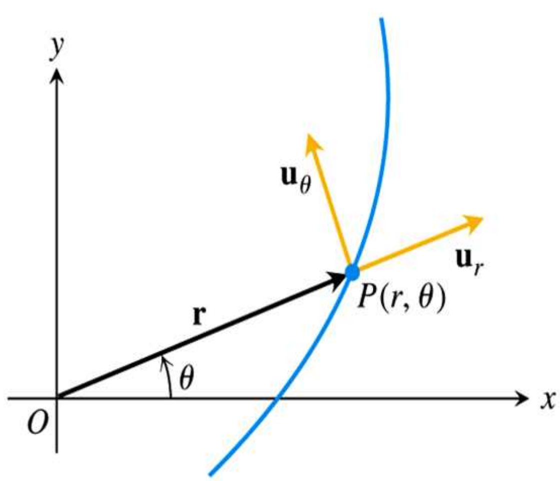


Figure 13.30, page 757

Note. We find from the above equations that

$$\begin{aligned}\frac{d\mathbf{u}_r}{d\theta} &= -(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j} = \mathbf{u}_\theta \\ \frac{d\mathbf{u}_\theta}{d\theta} &= -(\cos\theta)\mathbf{i} - (\sin\theta)\mathbf{j} = -\mathbf{u}_r.\end{aligned}$$

Differentiating $\mathbf{u}_r$ and $\mathbf{u}_\theta$ with respect to time t (and indicating derivatives with respect to time with dots, as physicists do), the Chain Rule gives

$$\dot{\mathbf{u}}_r = \frac{d\mathbf{u}_r}{d\theta}\dot{\theta} = \dot{\theta}\mathbf{u}_\theta, \qquad \dot{\mathbf{u}}_\theta = \frac{d\mathbf{u}_\theta}{d\theta}\dot{\theta} = -\dot{\theta}\mathbf{u}_r.$$

Note. With $\mathbf{r}$ as a position function, we can express velocity $\mathbf{v} = \dot{\mathbf{r}}$ as:

$$\mathbf{v} = \frac{d}{dt}[r\mathbf{u}_r] = \dot{r}\mathbf{u}_r + r\dot{\mathbf{u}}_r = \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta.$$

This is illustrated in the figure below.

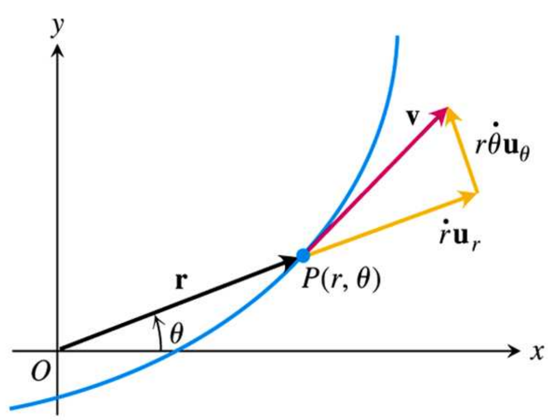


Figure 13.31, page 758

Note. We can express acceleration $\mathbf{a} = \dot{\mathbf{v}}$ as

$$\begin{aligned}\mathbf{a} &= (\ddot{r}\mathbf{u}_r + \dot{r}\dot{\mathbf{u}}_r) + (\dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta + r\dot{\theta}\dot{\mathbf{u}}_\theta) \\ &= (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta.\end{aligned}$$

Example. Page 760, number 4.

Definition. We introduce *cylindrical coordinates* by extending polar coordinates with the addition of a third axis, the z-axis, in a 3-dimensional right-hand coordinate system. The vector $\mathbf{k}$ is introduced as the direction vector of the z-axis.

Note. The position vector in cylindrical coordinates becomes $\mathbf{r} = r\mathbf{u}_r + z\mathbf{k}$. Therefore we have velocity and acceleration as:

$$\begin{aligned}\mathbf{v} &= \dot{r}\mathbf{u}_r + r\dot{\theta}\mathbf{u}_\theta + \dot{z}\mathbf{k} \\ \mathbf{a} &= (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta + \ddot{z}\mathbf{k}.\end{aligned}$$

The vectors $\mathbf{u}_r$, $\mathbf{u}_\theta$, and $\mathbf{k}$ make a right-hand coordinate system where

$$\mathbf{u}_r \times \mathbf{u}_\theta = \mathbf{k}, \quad \mathbf{u}_\theta \times \mathbf{k} = \mathbf{u}_r, \quad \mathbf{k} \times \mathbf{u}_r = \mathbf{u}_\theta.$$

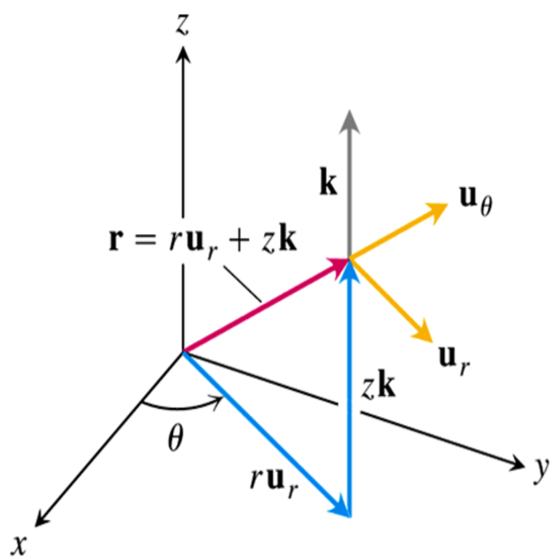


Figure 13.32, page 758

"Theorem." Newton's Law of Gravitation.

If $\mathbf{r}$ is the position vector of an object of mass m and a second mass of size M is at the origin of the coordinate system, then a (gravitational) force is exerted on mass m of

$$\mathbf{F} = -\frac{GmM}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|}.$$

The constant G is called the *universal gravitational constant* and (in terms of kilograms, Newtons, and meters) is 6.6726×10^{-11} Nm^{2}kg^{-2}.

Note. Newton's Second Law of Motion states that "force equals mass times acceleration" or, in the symbols above, $\mathbf{F} = m\ddot{\mathbf{r}}$. Combining this with Newton's Law of Gravitation, we get

$$m\ddot{\mathbf{r}} = -\frac{GmM}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|},$$

or

$$\ddot{\mathbf{r}} = -\frac{GM}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|}.$$

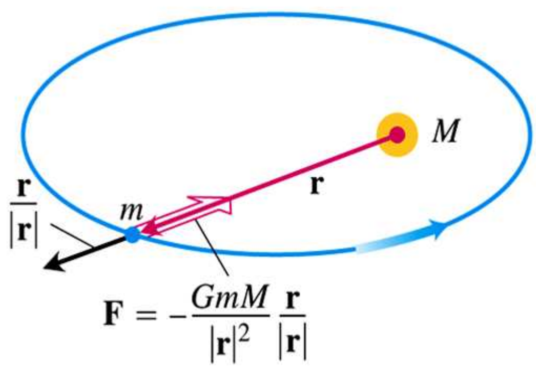


Figure 13.33, page 758

Note. Notice that $\ddot{\mathbf{r}}$ is a parallel (or, if you like, antiparallel) to $\mathbf{r}$, so $\mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0}$. This implies that

$$\frac{d}{dt}[\mathbf{r} \times \dot{\mathbf{r}}] = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0} + \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0}.$$

So $\mathbf{r} \times \dot{\mathbf{r}}$ must be a constant vector, say $\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{C}$. Notice that if $\mathbf{C} = \mathbf{0}$, then $\mathbf{r}$ and $\dot{\mathbf{r}}$ must be (anti)parallel and the motion of mass m must be in a line passing through mass M. This represents the case where mass m simply falls towards mass M and does not represent orbital motion, so we now assume $\mathbf{C} \neq \mathbf{0}$.

Lemma. If a mass M is stationary and mass m moves according to Newton's Law of Gravitation, then mass m will have motion which is restricted to a plane.

Proof. Since $\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{C}$, or more explicitly, $\mathbf{r}(t) \times \dot{\mathbf{r}}(t) = \mathbf{C}$ where $\mathbf{C}$ is a constant, then we see that the position vector $\mathbf{r}$ is always orthogonal to vector $\mathbf{C}$. Therefore $\mathbf{r}$ (in standard position) lies in a plane with $\mathbf{C}$ as its normal vector, and mass m is in this plane for all values of t. *Q.E.D.*

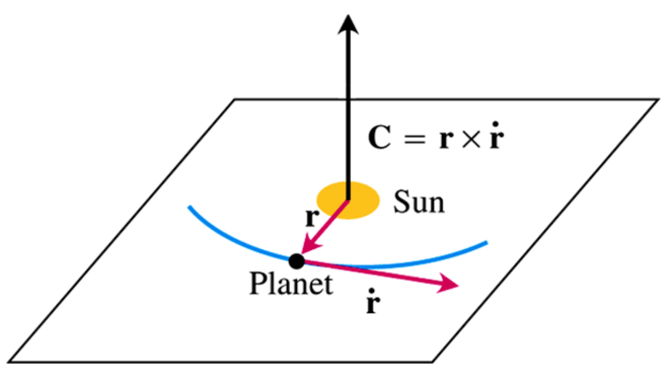


Figure 13.34, page 759

"Theorem." Kepler's First Law of Planetary Motion. Suppose a mass M is located at the origin of a coordinate system. Let mass m move under the influence of Newton's Law of Gravitation. Then m travels in a conic section with M at a focus of the conic.

Note. Kepler would think of mass M as the sun and mass m as one of the planets (each planet has an elliptical orbit). We can also think of mass m as an asteroid or comet in orbit about the sun (comets can have elliptic, parabolic, or hyperbolic orbits). It is also reasonable to think of mass M as the Earth and mass m as an object such as a satellite orbiting the Earth.

Proof of Kepler's First Law. The computations in this proof are based on work from *Celestial Mechanics* by Harry Pollard (*The Carus Mathematical Monographs*, Number 18, Mathematical Association of America, 1976). Let $\mathbf{r}(t) = \mathbf{r}$ be the position vector of mass m and let $r(t) = |\mathbf{r}(t)|$, or in shorthand notation $r = |\mathbf{r}|$. Then

$$\frac{d}{dt}\left[\frac{\mathbf{r}}{|\mathbf{r}|}\right] = \frac{d}{dt}\left[\frac{\mathbf{r}}{r}\right] = \frac{r\dot{\mathbf{r}} - \dot{r}\mathbf{r}}{r^2}$$

where the dots represents derivatives with respect to time t,

$$= \frac{r^2\dot{\mathbf{r}} - r\dot{r}\mathbf{r}}{r^3} = \frac{(\mathbf{r}\cdot\mathbf{r})\dot{\mathbf{r}} - (\mathbf{r}\cdot\dot{\mathbf{r}})\mathbf{r}}{r^3}$$

since $\dfrac{d}{dt}[r^2] = 2r\dot{r}$ by the Chain rule and $\dfrac{d}{dt}[r^2] = \dfrac{d}{dt}[\mathbf{r}\cdot\mathbf{r}] = 2\mathbf{r}\cdot\dot{\mathbf{r}}$, we have $r\dot{r} = \mathbf{r}\cdot\dot{\mathbf{r}}$,

$$= \frac{(\mathbf{r}\times\dot{\mathbf{r}})\times\mathbf{r}}{r^3}$$

since, in general, $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u}$ (see page 723, number 17).

That is, $\dfrac{d}{dt}\left[\dfrac{\mathbf{r}}{r}\right] = \dfrac{(\mathbf{r}\times\dot{\mathbf{r}})\times\mathbf{r}}{r^3} = \dfrac{\mathbf{C}\times\mathbf{r}}{r^3}$, or, multiplying both sides by $-Gm$,

$$-GM\frac{d}{dt}\left[\frac{\mathbf{r}}{r}\right] = \mathbf{C}\times\left(\frac{-GM}{r^3}\mathbf{r}\right)$$

or

$$GM\frac{d}{dt}\left[\frac{\mathbf{r}}{r}\right] = \mathbf{C}\times\left(\frac{GM}{r^3}\mathbf{r}\right). \qquad (*)$$

From Newton's Law of Gravitation and Newton's Second Law of Motion, we have $\ddot{\mathbf{r}} = \dfrac{-GM}{|\mathbf{r}|^2}\dfrac{\mathbf{r}}{|\mathbf{r}|} = \left(\dfrac{-GM}{r^3}\right)\mathbf{r}$, and so $(*)$ becomes

$$GM\frac{d}{dt}\left[\frac{\mathbf{r}}{r}\right] = \mathbf{C}\times(-\ddot{\mathbf{r}}) = \ddot{\mathbf{r}}\times\mathbf{C}. \qquad (**)$$

Integrate both sides of $(**)$ and add a constant vector of integration $\mathbf{e}$ to get

$$GM\left(\frac{\mathbf{r}}{r}+\mathbf{e}\right) = \dot{\mathbf{r}}\times\mathbf{C} \qquad (***)$$

(remember $\mathbf{C}$ is constant). Dotting both sides of $(***)$ with $\mathbf{r}$ gives

$$GM\left(\mathbf{r}\cdot\frac{\mathbf{r}}{r}+\mathbf{r}\cdot\mathbf{e}\right) = (\dot{\mathbf{r}}\times\mathbf{C})\cdot\mathbf{r}$$

or

$$GM\left(\frac{|\mathbf{r}|^2}{r}+\mathbf{r}\cdot\mathbf{e}\right)=(\mathbf{r}\times\dot{\mathbf{r}})\cdot\mathbf{C}$$

by a property of the triple scalar product (see page 704), or

$$GM(r+\mathbf{r}\cdot\mathbf{e})=\mathbf{C}\cdot\mathbf{C}=C^2$$

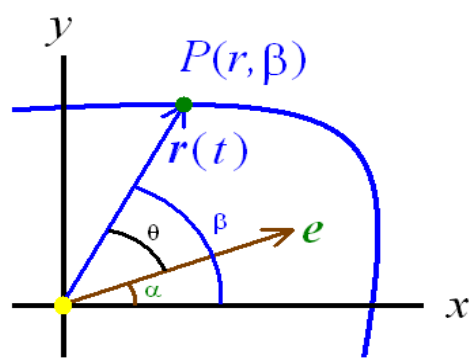
where $C=|\mathbf{C}|$, or

$$r+\mathbf{r}\cdot\mathbf{e}=\frac{C^2}{GM}. \qquad (****)$$

As commented above, if $\mathbf{C}=\mathbf{0}$ then we have motion along a line towards mass M at the origin, so we assume $\mathbf{C}\neq\mathbf{0}$. Finally, we interpret $e=|\mathbf{e}|$. First, suppose $e=0$. Then $r=C^2/(GM)$ (a constant) and so the motion is circular about central mass M. Recall that a circle is a conic section of eccentricity 0. Second, suppose $e\neq 0$. From $(***)$, $GM\left(\frac{\mathbf{r}}{r}+\mathbf{e}\right)=\dot{\mathbf{r}}\times\mathbf{C}$ where $\mathbf{C}=\mathbf{r}\times\dot{\mathbf{r}}$. By properties of the cross product, $\frac{\mathbf{r}}{r}+\mathbf{e}$ and $\mathbf{r}$ are both orthogonal to $\mathbf{C}$. Therefore $\mathbf{r}\cdot\mathbf{C}=0$ and

$$\left(\frac{\mathbf{r}}{r}+\mathbf{e}\right)\cdot\mathbf{C}=\frac{1}{r}(\mathbf{r}\cdot\mathbf{C})+\mathbf{e}\cdot\mathbf{C}=0+\mathbf{e}\cdot\mathbf{C}=\mathbf{e}\cdot\mathbf{C}=0.$$

So $\mathbf{e}$ is orthogonal to $\mathbf{C}$. Since $\mathbf{C}$ is orthogonal to the plane of motion, then $\mathbf{e}$ lies in the plane of motion (when put in standard position). Introduce vector $\mathbf{e}$ in the plane of motion (say the xy-plane) and let α be the angle between the positive x-axis and $\mathbf{e}$. Let $\mathbf{r}(t)$ be in standard position and represent the head of $\mathbf{r}(t)$ as $P(r, \beta)$ in polar coordinates r and β. Define θ as $\beta - \alpha$:



The relationship between $\mathbf{r}$, $\mathbf{e}$, α, β, and θ.

Then $\mathbf{r} \cdot \mathbf{e} = re\cos\theta$. So equation $(****)$ gives

$$r + \mathbf{r} \cdot \mathbf{e} = \frac{C^2}{GM} \text{ or } r + re\cos\theta = \frac{C^2}{GM} \text{ or } r = \frac{C^2/(GM)}{1 + e\cos\theta}.$$

This is a conic section of eccentricity e in polar coordinates (r, θ) (see page 668). Notice that r is a minimum when the denominator is largest. This occurs when $\theta = 0$ and gives $r_0 = \dfrac{C^2/(GM)}{1 + e}$. We can solve for C^2 to get $C^2 = GM(1 + e)r_0$. Therefore the motion is described in terms of e and r_0 as $r = \dfrac{(1 + e)r_0}{1 + e\cos\theta}$. For (noncircular) orbits about the sun, r_0 is called the *perihelion* distance (if the Earth is the central mass, r_0 is the *perigee* distance). In conclusion, the motion of mass m is a conic section of eccentricity e and is described in polar coordinates (r, θ) as $r = \dfrac{(1 + e)r_0}{1 + e\cos\theta}$. *Q.E.D.*

Theorem. Kepler's Second Law of Planetary Motion.

Suppose a mass M is located at the origin of a coordinate system and that mass m move according to Kepler's First Law of Planetary Motion. Then the radius vector from mass M to mass m sweeps out equal areas in equal times.

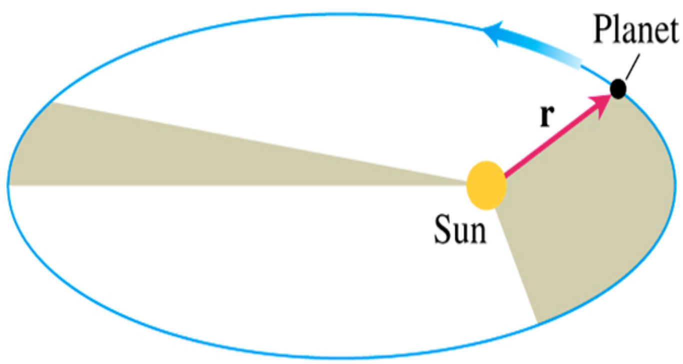


Figure 13.35, page 759

Note. If we know the orbit of an object (that is, if we know the conic section from Kepler's First Law which describes the objects position), then Kepler's Second Law allows us to find the location of the object at any given time (assuming we have some initial position from which time is measured).

Proof of Kepler's Second Law. In Lemma we have seen that the vector $\mathbf{r}(t) \times \dot{\mathbf{r}}(t) = \mathbf{C}$ is a constant. If we express the position vector in polar coordinates, we get $\mathbf{r}(t) = \mathbf{r} = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j}$. Therefore $\dot{\mathbf{r}}(t) = (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\mathbf{i} + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)\mathbf{j}$. We also know that $\mathbf{C} = C\mathbf{k}$. So the equation $\mathbf{r}(t) \times \dot{\mathbf{r}}(t) = \mathbf{C}$ yields

$$\mathbf{r}(t) \times \dot{\mathbf{r}}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r\cos\theta & r\sin\theta & 0 \\ \dot{r}\cos\theta - r\dot{\theta}\sin\theta & \dot{r}\sin\theta + r\dot{\theta}\cos\theta & 0 \end{vmatrix}$$

$$= \Big\{(r\cos\theta(\dot{r}\sin\theta + r\dot{\theta}\cos\theta)) - r\sin\theta(\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\Big\}\,\mathbf{k}$$

$$= (r\dot{r}\cos\theta\sin\theta + r^2\dot{\theta}\cos^2\theta - r\dot{r}\sin\theta\cos\theta + r^2\dot{\theta}\sin^2\theta)\mathbf{k}$$

$$= (r^2\dot{\theta})\mathbf{k} = C\mathbf{k}.$$

Now in polar coordinates, area is calculated as $A = \int_a^b \frac{1}{2}r^2(\theta)d\theta$ and so the derivative of area with respect to time is (by the Chain Rule and the Fundamental Theorem of Calculus Part I) $\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = \frac{1}{2}r^2(\theta)\dot{\theta} = \frac{1}{2}r^2\dot{\theta}$. Therefore $\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{1}{2}C$ where C is constant. Hence the rate of change of time is constant and the radius vector sweeps out equal areas in equal times. *Q.E.D.*

Theorem. Kepler's Third Law of Planetary Motion.

Suppose a mass M is located at the origin of a coordinate system and that mass m move according to Kepler's First Law of Planetary Motion and that the orbit is a circle or ellipse. Let T be the time it takes for mass m to compete one orbit of mass M and let a be the semimajor axis of the elliptical orbit (or the radius of the circular orbit). Then, $\dfrac{T^2}{a^3} = \dfrac{4\pi^2}{GM}$.

Note. Kepler's Third Law allows us to find a relationship between the *orbital period* T of a planet and the size of the planet's orbit. For example, the semimajor axis of the orbit of Mercury is 0.39 AU and the orbital period of Mercury is 88 days. The semimajor axis of the orbit of the Earth is 1 AU and the orbital period is 365.25 days. The semimajor axis of Neptune is 30.06 AU and the orbital period is 60,190 days (165 Earth years).

Proof of Kepler's Third Law. The area of the ellipse which describes the orbit is

$$A = \int_0^T \left(\frac{dA}{dt}\right) dt = \int_0^T \frac{1}{2}C\,dt = \frac{1}{2}CT$$

since $dA/dt = \frac{1}{2}C$ from the proof of Kepler's Second Law. The area of an ellipse with semimajor axis of length a and semiminor axis of length b is

πab. Therefore

$$\frac{1}{2}CT = \pi ab = \pi a^2\sqrt{1-e^2}$$

since from page 666 $e = \dfrac{a^2-b^2}{a}$ or $e^2a^2 = a^2-b^2$ or $b^2 = a^2(1-e^2)$ or $b = a\sqrt{1-e^2}$. Therefore

$$\frac{C^2T^2}{4} = \pi^2a^4(1-e^2)$$

or

$$\frac{T^2}{a^3} = \frac{4\pi^2(1-e^2)a}{C^2}. \qquad (*)$$

Now the maximum value of r, denoted $r_{\max}$, occurs when $\theta = \pi$: $r_{\max} = \left.\dfrac{(1+e)r_0}{1+e\cos\theta}\right|_{\theta=\pi} = \dfrac{r_0(1+e)}{1-e}$. Since $2a = r_0 + r_{\max}$, then

$$a = \frac{r_0 + \frac{r_0(1+e)}{1-e}}{2} = \frac{r_0(1-e)+r_0(1+e)}{2(1-e)} = \frac{r_0}{1-e}.$$

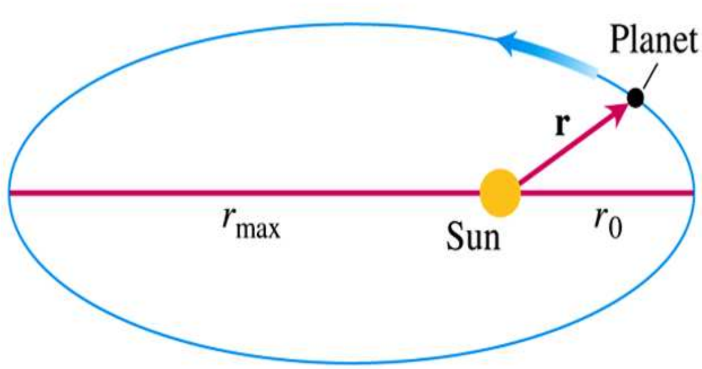


Figure 13.36, page 760

Substituting this value of a into $(*)$ give

$$\frac{T^2}{a^3} = \frac{4\pi^2(1-e^2)}{C^2}\frac{r_0}{1-e} = \frac{4\pi^2(1-e^2)}{GM(1+e)r_0}\frac{r_0}{1-e}$$

since $C^2 = GM(1+e)r_0$ from the proof of Kepler's First Law. Therefore $\dfrac{T^2}{a^3} = \dfrac{4\pi^2}{GM}$. *Q.E.D.*

A Historical Note. Claudius Ptolemy (90 CE–168 CE) presented a model of the universe which was widely accepted for almost 1400 years. In his *Almagest* he proposed that the universe had the Earth in the center with the planets Mercury, Venus, Mars, Jupiter, and Saturn, along with the sun and moon, orbiting around the Earth once every day in circular orbits. In addition, the stars were located on a sphere which rotated once a day. His model was quite complicated and required a number of "epicycles" which were additional circles needed to explain the complicated observed motion of the planets (in particular, the occasional retrograde movement seen in the motion of the superior planets Mars, Jupiter, and Saturn). Some of these ideas were inherited from Ptolemy's predecessors such as Hipparchus and Apollonius of Perga (both living around 200 BCE). Surprisingly, another ancient Greek astronomer which predates each of these, Aristarchus of Samos (310 BCE–230 BCE), proposed that the Sun is the center of the universe and that the Earth is a planet orbiting the sun, just like each of the other five planets. However, Ptolemy's model was much more widely accepted and adopted by the Christian church.

Polish astronomer Nicolaus Copernicus (1473–1543) proposed again that the sun is the center of the universe and that the planets move in perfect circles around the sun with the sun at the center of the circles. His ideas were published shortly before his death in *De revolutionibus orbium coelestium* (*On the Revolutions of the Celestial Spheres*). The *Copernican system* became synonymous with heliocentrism. Copernicus's model was meant to simplify the complicated model of Ptolemy, yet its predictive power was not as strong as that of Ptolemy's model (due to the fact that Copernicus insisted on circular orbits).

Johannes Kepler (1571–1630) used extremely accurate observational data of planetary positions (he used the data of Tycho Brahe which was entirely based on naked eye observations) to discover his laws of planetary motion. After years of trying, he used data, primarily that of the position of Mars, to fit an ellipse to the data. In 1609 he published his first two laws in *Astronomia nova* (*A New Astronomy*). However, Kepler's work was not based on any particular theoretical framework, but only on the observations. It would require another to actually *explain* the motion of the planets.

Isaac Newton (1643–1727) invented calculus in 1665 and 1666, but failed to publish it at the time (which lead to years of controversy with Gottfried Leibniz). In 1684, Edmund Halley (of the comet fame) asked Newton what type of path an object would follow under an inverse-square law of attraction (of gravity). Newton immediately replied the shape was an ellipse and that he had worked it out years before (but not published it). Halley was so impressed, he convinced Newton to write up his ideas and Newton published *Principia Mathematica* in 1687 (with Halley covering the expense of publication). This book was the invention of classical physics and is sometimes called the greatest scientific book of all time! Newton's proof of Kepler's laws from his inverse-square law of gravitation is certainly one of the greatest accomplishments of classical physics and Newton's techniques (which we have seen in this section) ruled physics until the time of Albert Einstein (1879–1955)!