Chapter 14. Partial Derivatives

14.4. The Chain Rule

Note. We now wish to find derivatives of functions of several variables when the variables themselves are functions of additional variables. That is, we want to deal with compositions of functions of several variables. This requires Chain Rules.

Theorem 5. Chain Rule for Functions of Two Independent Variables.

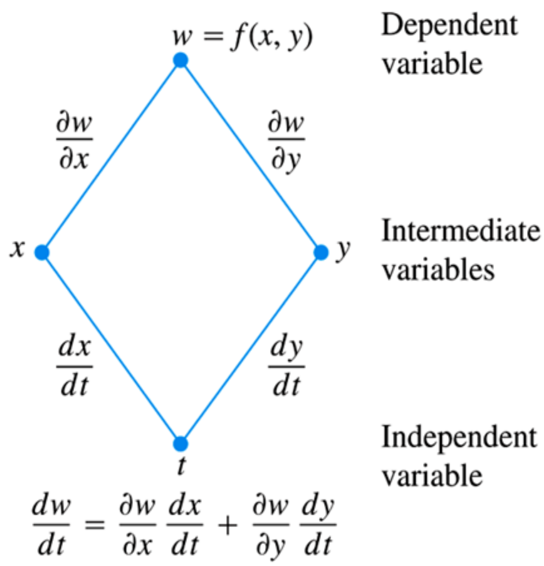
If $w = f(x, y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dw}{dt} = f_x(x(t), y(t))[x'(t)] + f_y(x(t), y(t))[y'(t)],$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\left[\frac{dx}{dt}\right] + \frac{\partial f}{\partial y}\left[\frac{dy}{dt}\right] = \frac{\partial w}{\partial x}\left[\frac{dx}{dt}\right] + \frac{\partial w}{\partial y}\left[\frac{dy}{dt}\right].$$

Note. We will remember the various versions of the Chain Rule which we address in this section using "branch diagrams" which reflect the relationships between each of the variables. For example, Theorem 5 can be illustrated as:



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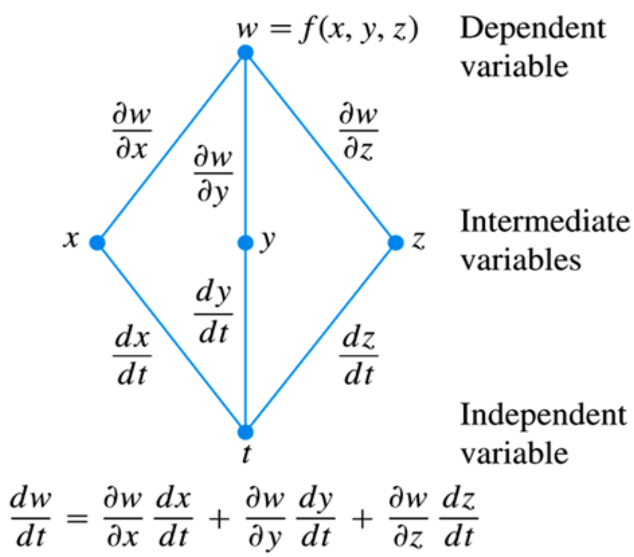
Example. Page 800, number 4.

Theorem 6. Chain Rule for Functions of Three Independent Variables.

If $w = f(x, y, z)$ is differentiable and x, y, and z are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\left[\frac{dx}{dt}\right] + \frac{\partial w}{\partial y}\left[\frac{dy}{dt}\right] + \frac{\partial w}{\partial z}\left[\frac{dz}{dt}\right].$$



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Example. Page 800, number 34.

Note. To motivate other function compositions, the textbook describes the following. Suppose we are interested in the temperature $w = f(x, y, z)$ at points (x, y, z) on the earth's surface, we might prefer to think of x, y, and z as functions of the variables r and s that give the points' longitudes and latitudes. If $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$, we could then express the temperature as a function of r and s with the composite function $w = f(g(r, s), h(r, s), k(r, s))$. Therefore w has partial derivatives with respect to r and s, as given in the following theorem.

Theorem 7. Chain Rule for Two Independent Variables and Three Intermediate Variables.

Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s given by the formulas:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\left[\frac{\partial x}{\partial r}\right] + \frac{\partial w}{\partial y}\left[\frac{\partial y}{\partial r}\right] + \frac{\partial w}{\partial z}\left[\frac{\partial z}{\partial r}\right]$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\left[\frac{\partial x}{\partial s}\right] + \frac{\partial w}{\partial y}\left[\frac{\partial y}{\partial s}\right] + \frac{\partial w}{\partial z}\left[\frac{\partial z}{\partial s}\right].$$

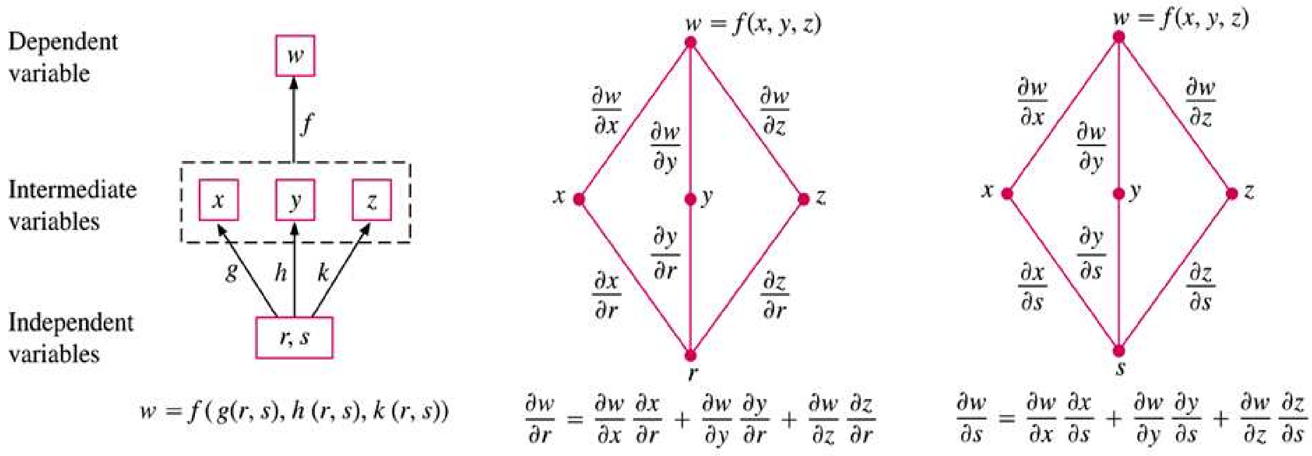

Figure 14.21, Page 796

Example. Page 800, number 16.

Note. Implicit Differentiation Revisited.

The two-variable Chain Rule in Theorem 5 leads to a formula that takes some of the algebra out of implicit differentiation. Suppose that

1. The function $F(x, y)$ is differentiable and

2. The equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, say $y = h(x)$.

Since $w = F(x, y) = 0$, the derivative dw/dx must be zero. Computing the derivative from the Chain Rule (see Figure 14.24 below), we find

$$\begin{aligned} 0 &= \frac{dw}{dx} = F_x \frac{dx}{dx} + F_y \frac{dy}{dx} \\ &= F_x + F_y \frac{dy}{dx}. \end{aligned}$$

If $F_y = \partial w/\partial y \neq 0$, we can solve this equation for dy/dx to get $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$. In summary, we have the following theorem.

Theorem 8. A Formula for Implicit Differentiation.

Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x. That at any point where $F_y \neq 0$, $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$.

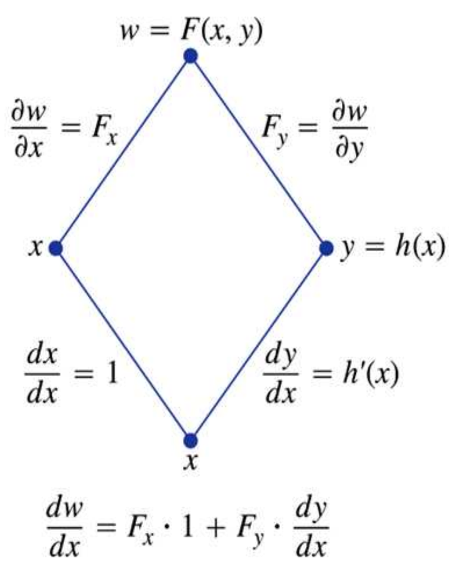

Figure 14.24, Page 798

Example. Page 800, Number 28.

Note. Theorem 8 can be extended to three variables. Suppose that the equation $F(x, y, z) = 0$ defines the variable z implicitly as a function $z = f(x, y)$. Then partial derivatives of z with respect to z and y are (when $F_z \neq 0$) given by:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

Example. Page 801, number 44.

Note. We have claimed that we would address functions of *several* variables, but have really only concentrated on functions of two or three variables. Suppose that $w = f(x, y, \ldots, v)$ is a differentiable function of the variables $x, y, \ldots, v$ (a finite number of variables) and that $x, y, \ldots, v$ are differentiable functions of $p, q, \ldots, t$ (a finite number of variables). Then w is a differentiable function of the variables $p, q, \ldots, t$ and the partial derivatives of w with respect to these variables are given by equations of the form:

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x}\left[\frac{\partial x}{\partial p}\right] + \frac{\partial w}{\partial y}\left[\frac{\partial y}{\partial p}\right] + \cdots + \frac{\partial w}{\partial v}\left[\frac{\partial v}{\partial p}\right].$$