Chapter 14. Partial Derivatives

14.6. Tangent Planes and Differentials

Note. If \( \mathbf{r} = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k} \) is a smooth curve on the level surface \( f(x, y, z) = c \) of a differentiable function \( f \), then \( f(g(t), h(t), k(t)) = c \). Differentiating both sides of this equation with respect to \( t \) gives

\[
\frac{d}{dt}[f(g(t), h(t), k(t))] = \frac{d}{dt}[c]
\]

\[
\frac{\partial f}{\partial x} \left[ \frac{dg}{dt} \right] + \frac{\partial f}{\partial y} \left[ \frac{dh}{dt} \right] + \frac{\partial f}{\partial z} \left[ \frac{dk}{dt} \right] = 0
\]

\[
\left( \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \right) \cdot \left( \frac{dg}{dt} \mathbf{i} + \frac{dh}{dt} \mathbf{j} + \frac{dk}{dt} \mathbf{k} \right) = 0
\]

\[
(\nabla f) \cdot \left( \frac{d\mathbf{r}}{dt} \right) = 0.
\]

At every point along the curve, \( \nabla f \) is orthogonal to the curve’s velocity vector. In the figure below, we see that all the velocity vectors at point \( P_0 \) are orthogonal to \( \nabla f \) at \( P_0 \), so the curves’ tangent lines all lie in the plane through \( P_0 \) normal to \( \nabla f \). Therefore, the gradient of \( f \) at \( P_0 \) will
act as a normal vector to the tangent plane to the surface at $P_0$.

![Figure 14.32, Page 810]

**Definition.** The *tangent plane* at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function $f$ is the plane through $P_0$ normal to $\nabla f|_{P_0}$. The *normal line* of the surface at $P_0$ is the line through $P_0$ parallel to $\nabla f|_{P_0}$.

**Note.** The equation of the tangent plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$ is

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0.$$ 

The equation of the normal line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$ is

$$x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t.$$
Example. Page 817, number 4.

Note. If we consider the function \( z = f(x, y) \), then the tangent plane to this surface at the point \((x_0, y_0, f(x_0, y_0))\) is

\[
f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.
\]

Example. Page 817, number 10.

Note. We now use differentials to estimate changes in functions, similar to what was done for functions of a single variable in section 3.11. To estimate the change in the value of a differentiable function \( f \) when we move a small distance \( ds \) from a point \( P_0 \) in a particular direction \( \mathbf{u} \), we use the differential

\[
df = \left( \nabla f |_{P_0} \cdot \mathbf{u} \right) ds.
\]

Notice that \( df \) is the directional derivative of \( f \) times the distance increment \( ds \).

Example. Page 817, number 20.
Definition. The linearization of a function $f(x, y)$ at a point $(x_0, y_0)$ where $f$ is differentiable is the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

The approximation $f(x, y) \approx L(x, y)$ is the standard linear approximation of $f$ at $(x_0, y_0)$.

Note. In fact, the plane $z = L(x, y)$ is tangent to the surface $z = f(x, y)$ at the point $(x_0, y_0)$ (just as the line $y = L(x)$ was the tangent line to $y = f(x)$ at the point of approximation in section 3.11). Thus, the linearization of a function of two variables is a tangent-plane approximation. As long as $(x, y)$ is “close to” $(x_0, y_0)$ (that is, if $\Delta x$ and $\Delta y$ are small), then $L(x, y)$ will take on approximately the same values as $f(x, y)$.

Example. Page 818, number 30.

Note. If $f$ has continuous first and second partial derivatives throughout an open set containing a rectangle $R$ centered at $(x_0, y_0)$ and if $M$ is any upper bound for the values of $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on $R$, then the error $E(x, y)$ incurred in replacing $f(x, y)$ on $R$ by its linearization

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$
satisfies the inequality  

$$|E(x, y)| \leq \frac{1}{2}M(|x - x_0| + |y - y_0|)^2.$$  

Notice that the error is small when $M$, $\Delta x$, and/or $\Delta y$ are small (especially $\Delta x$ and $\Delta y$).

**Example.** Page 818, number 50.

**Definition.** The *differentials* $dx$ and $dy$ are independent variables (so they can take on any values). Often we take $dx = \Delta x = x - x_0$ and $dy = \Delta y = y - y_0$. If we move from $(x_0, y_0)$ to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change  

$$df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$  

in the linearization of $f$ is called the *total differential* of $f$.

**Example.** Page 819, number 52.
Note. We can extend the ideas of this section to functions of more than two variables. For functions of three variables, we have the following.

1. The linearization of \( f(x, y, z) \) at a point \( P_0(x_0, y_0, z_0) \) is

\[
L(x, y, z) = f(P_0) + f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0).
\]

2. Suppose that \( R \) is a closed rectangular solid centered at \( P_0 \) and lying in an open region on which the second partial derivatives of \( f \) are continuous. Suppose also that \( |f_{xx}|, |f_{yy}|, |f_{zz}|, |f_{xy}|, |f_{xz}|, \) and \( |f_{yz}| \) are all less than or equal to \( M \) throughout \( R \). Then the error

\[
E(x, y, z) = f(x, y, z) - L(x, y, z)
\]

in the approximation of \( f \) by \( L \) is bounded throughout \( R \) by

\[
|E| \leq \frac{1}{2}M(|x - x_0| + |y - y_0| + |z - z_0|)^2.
\]

3. If the second partial derivatives of \( f \) are continuous and if \( x, y, \) and \( z \) change from \( x_0, y_0, \) and \( z_0 \) by “small” amounts \( dx, dy, \) and \( dz, \) the total differential

\[
df = f_x(P_0)dx + f_y(P_0)dy + f_z(P_0)dz
\]

gives a “good” approximation of the resulting change in \( f \).

Example. Page 818, number 44.