Chapter 15. Multiple Integrals

15.1. Double and Iterated Integrals over Rectangles

Note. In this section we extend the idea of integral to functions of two variables $f(x, y)$ over a bounded rectangle R in the plane.

Definition. Let $f(x, y)$ be a function defined on a rectangular region $R = \{(x, y) \mid x \in [a, b], y \in [c, d]\}$. Subdivide R into small rectangles using a network of lines parallel to the x- and y-axes. The lines divide R into n rectangular pieces, where the number of pieces n gets large as the width and height of each piece gets small. These rectangles form a *partition* of R. A small rectangular piece of width Δx and height Δy has area $\Delta A = \Delta x \Delta y$. If we number the small pieces partitioning R in some order, then their areas are given by numbers $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$, where

ΔA_k is the area of the kth rectangle.

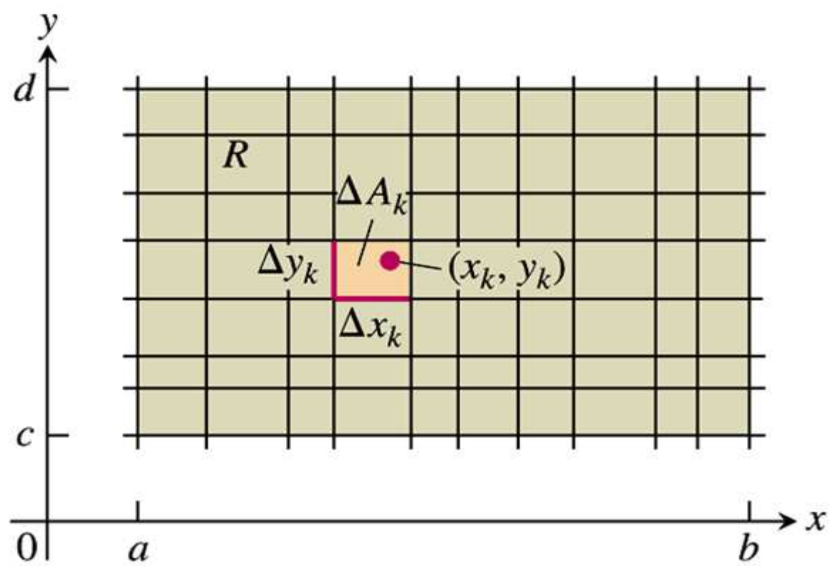

Figure 15.1, Page 854

Definition. To form a *Riemann sum* over R, we choose a point (x_k, y_k) in the kth small rectangle, multiply the value of f at the point by the area ΔA_k and add together the products:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k.$$

Depending on how we pick (x_k, y_k) in the kth small rectangle, we get different values for S_n.

Note. A Riemann sum is a "good" approximation of the volume above R and below $z = f(x, y)$ when the ΔA's are small.

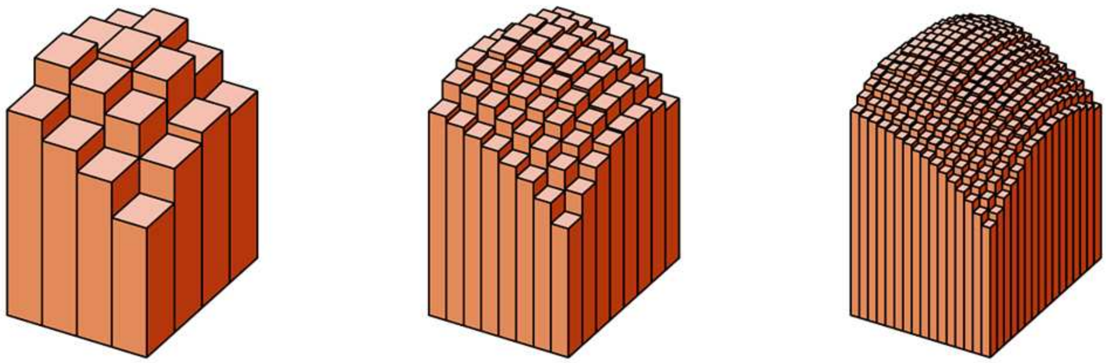

Figure 15.3, Page 856

Definition. The *norm* of a partition P, denoted $\|P\|$, is the largest width or height of any rectangle in the partition:

$$\|P\| = \max_{1 \leq k \leq n} \{\Delta x_k, \Delta y_k\}.$$

If the limit

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k$$

exists and is the same regardless of how the partition and (x_k, y_k) are chosen, then f is *integrable* over R and the value of the limit is the *double integral* of f over R, denoted:

$$\iint_R f(x, y)\, dA = \iint_R f(x, y)\, dx\, dy.$$

Theorem. If $f(x, y)$ is continuous on rectangular region R, then f is integrable over R.

Note. When $f(x, y)$ is a nonnegative function over a rectangular region R in the xy-plane, we may interpret the double integral of f over R as the volume of the 3-dimensional solid region over the xy-plane bounded below by R and above by the surface $z = f(x, y)$. In fact, we take this as the definition of such a volume.

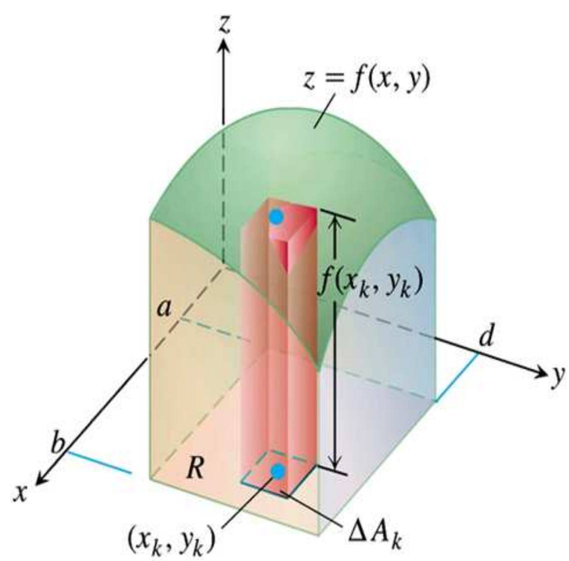

Figure 15.2, Page 855

Theorem 1. Fubini's Theorem (First Form)

If $f(x, y)$ is continuous throughout the rectangular region $R = \{(x, y) \mid x \in [a, b], y \in [c, d]\}$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

The second two integrals are called *iterated integrals*.

Note. Fubini's Theorem allows us to evaluate double integrals by integrating with respect to one variable at a time. This means that when we calculate a volume by "slices" (slices are really differentials), we may start with either dx-slices or dy-slices.

Examples. Page 858, number 6. Page 859, numbers 16 and 28.

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