Chapter 15. Multiple Integrals

15.1. Double and Iterated Integrals over Rectangles

Note. In this section we extend the idea of integral to functions of two variables \( f(x, y) \) over a bounded rectangle \( R \) in the plane.

Definition. Let \( f(x, y) \) be a function defined on a rectangular region \( R = \{(x, y) \mid x \in [a, b], y \in [c, d]\} \). Subdivide \( R \) into small rectangles using a network of lines parallel to the \( x \)- and \( y \)-axes. The lines divide \( R \) into \( n \) rectangular pieces, where the number of pieces \( n \) gets large as the width and height of each piece gets small. These rectangles form a partition of \( R \). A small rectangular piece of width \( \Delta x \) and height \( \Delta y \) has area \( \Delta A = \Delta x \Delta y \). If we number the small pieces partitioning \( R \) in some order, then their areas are given by numbers \( \Delta A_1, \Delta A_2, \ldots, \Delta A_n \), where
\( \Delta A_k \) is the area of the \( k \)th rectangle.

![Figure 15.1, Page 854](image)

**Definition.** To form a *Riemann sum* over \( R \), we choose a point \((x_k, y_k)\) in the \( k \)th small rectangle, multiply the value of \( f \) at the point by the area \( \Delta A_k \) and add together the products:

\[
S_n = \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k.
\]

Depending on how we pick \((x_k, y_k)\) in the \( k \)th small rectangle, we get different values for \( S_n \).
**Note.** A Riemann sum is a “good” approximation of the volume above \( R \) and below \( z = f(x, y) \) when the \( \Delta A \)’s are small.

![Figure 15.3, Page 856](image)

**Definition.** The *norm* of a partition \( P \), denoted \( \|P\| \), is the largest width or height of any rectangle in the partition:

\[
\|P\| = \max_{1 \leq k \leq n} \{ \Delta x_k, \Delta y_k \}.
\]

If the limit

\[
\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k
\]

exists and is the same regardless of how the partition and \((x_k, y_k)\) are chosen, then \( f \) is *integrable* over \( R \) and the value of the limit is the *double integral* of \( f \) over \( R \), denoted:

\[
\int \int_{R} f(x, y) \, dA = \int \int_{R} f(x, y) \, dx \, dy.
\]
**Theorem.** If \( f(x, y) \) is continuous on rectangular region \( R \), then \( f \) is integrable over \( R \).

**Note.** When \( f(x, y) \) is a nonnegative function over a rectangular region \( R \) in the \( xy \)-plane, we may interpret the double integral of \( f \) over \( R \) as the volume of the 3-dimensional solid region over the \( xy \)-plane bounded below by \( R \) and above by the surface \( z = f(x, y) \). In fact, we take this as the definition of such a volume.
Theorem 1. Fubini’s Theorem (First Form)

If \( f(x, y) \) is continuous throughout the rectangular region \( R = \{(x, y) \mid x \in [a, b], y \in [c, d]\} \), then

\[
\int \int_R f(x, y) \, dA = \int_c^d \int_a^b f(x, y) \, dx \, dy = \int_a^b \int_c^d f(x, y) \, dy \, dx.
\]

The second two integrals are called \textit{iterated integrals}.

\textbf{Note.} Fubini’s Theorem allows us to evaluate double integrals by integrating with respect to one variable at a time. This means that when we calculate a volume by “slices” (slices are really differentials), we may start with either \( dx \)-slices or \( dy \)-slices.