Chapter 15. Multiple Integrals

15.2. Double Integrals over General Regions

Note. Let $R$ be a non-rectangular region in the plane. A partition of $R$ is formed in a manner similar to rectangular regions, but we now only take rectangles which lie entirely inside region $R$ (see Figure 15.8 below). As before, we number the rectangles and let $\Delta A_k$ be the area of the $k$th rectangle. Choose a point $(x_k, y_k)$ in the $k$th rectangle and compute a Riemann sum as

$$S_n = \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k.$$

Again, we define the double integral of $f(x, y)$ over $R$ as

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k) \Delta A_k = \int \int_{R} f(x, y) \, dA.$$
Definition. When $f(x, y)$ is a positive function over a region $R$ in the $xy$-plane, we define the volume bounded below by $R$ and above by the surface $z = f(x, y)$ to be the double integral of $f$ over $R$.

Theorem 2. Fubini’s Theorem (Stronger Form)

Let $f(x, y)$ be continuous on a region $R$.

1. If $R$ is defined by $x \in [a, b]$, $g_1(x) \leq y \leq g_2(x)$, with $g_1$ and $g_2$ continuous on $[a, b]$, then

$$
\int \int_R f(x, y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx.
$$
2. If \( R \) is defined by \( y \in [c, d] \), \( h_1(y) \leq x \leq h_2(y) \), with \( h_1 \) and \( h_2 \) continuous on \([c, d]\), then

\[
\int \int_R f(x, y) \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy.
\]

Figures 15.10 and 15.11, Pages 860 and 861

**Example.** Page 866, number 20.

**Note. Using Vertical Cross-Sections.**

When faced with evaluating \( \int \int_R f(x, y) \, dA \), integrating first with respect to \( y \) and then with respect to \( x \), do the following three steps:

1. **Sketch.** Sketch the region of integration and label the bounding curves.
2. Find the $y$-limits of integration. Imagine a vertical line $L$ cutting through $R$ in the direction of increasing $y$. Mark the $y$-values where $L$ enters and leaves. These are the $y$-limits of integration and are usually functions of $x$ (instead of constants).

3. Find the $x$-limits of integration. Choose $x$-limits that include all the vertical lines through $R$. The integral shown below is

$$
\int \int_R f(x, y) \, dA = \int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x, y) \, dy \, dx.
$$

Using Horizontal Cross-Sections.

To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in Steps 2 and 3. The integral below is

$$
\int \int_R f(x, y) \, dA = \int_{y=0}^{y=1} \int_{x=1-y}^{x=\sqrt{1-y^2}} f(x, y) \, dx \, dy.
$$

Figures 15.14 and 15.15, Page 863
Examples. Page 866, numbers 40 and 50.


If \( f(x, y) \) and \( g(x, y) \) are continuous on the bounded region \( R \), then the following properties hold.

1. **Constant Multiple:** \( \int \int_{R} cf(x, y) \, dA = c \int \int_{R} f(x, y) \, dA \) for any constant \( c \)

2. **Sum and Difference:** \( \int \int_{R} (f(x, y) \pm g(x, y)) \, dA = \int \int_{A} f(x, y) \, dA \pm \int \int_{R} g(x, y) \, dA \)

3. **Domination:**
   
   (a) \( \int \int_{R} f(x, y) \, dA \geq 0 \) if \( f(x, y) \geq 0 \) on \( R \)
   
   (b) \( \int \int_{R} f(x, y) \, dA \geq \int \int_{R} g(x, y) \, dA \) if \( f(x, y) \geq g(x, y) \) on \( R \)

4. **Additivity:** \( \int \int_{R} f(x, y) \, dA = \int \int_{R_1} f(x, y) \, dA + \int \int_{R_2} f(x, y) \, dA \)
   if \( R \) is the union of two non-overlapping regions \( R_1 \) and \( R_2 \)

Examples. Page 866, number 58 and Page 867, number 76.