Chapter 15. Multiple Integrals

15.4. Double Integrals in Polar Form

Note. Suppose that a function \( f(r, \theta) \) is defined over a region \( R \) that is bounded by the rays \( \theta = \alpha \) and \( \theta = \beta \) and the continuous curves \( r = g_1(\theta) \) and \( r = g_2(\theta) \). Suppose also that \( 0 \leq g_1(\theta) \leq g_2(\theta) \leq a \) for every value of \( \theta \) between \( \alpha \) and \( \beta \). Then \( R \) lies in a fanshaped region \( Q \) defined by \( \{(r, \theta) \mid r \in [0, a], \theta \in [\alpha, \beta]\} \).

![Figure 15.21, Page 871](image)

Note. We cover \( Q \) by a grid of a circular arcs and rays. The arcs are cut from circles centered at the origin, with radii \( \Delta r, 2\Delta r, \ldots, m\Delta r \), where \( \Delta r = a/m \). The rays are given by:

\[
\theta = \alpha, \theta = \alpha + \Delta \theta, \theta = \alpha + 2\Delta \theta, \ldots, \theta = \alpha + m'\Delta \theta = \beta
\]
where $\Delta \theta = (\beta - \alpha)/m'$. The arcs and rays partition $Q$ into small patches called “polar rectangles.” We number the polar rectangles that lie inside $R$, calling their areas $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$. We let $(r_k, \theta_k)$ be any point in the polar rectangle whose area is $\Delta A_k$. We then form the sum $S_n = \sum_{k=1}^{n} f(r_k, \theta_k) \Delta A_k$. If $f$ is continuous throughout $R$, this sum will approach a limit as we refine the grid to make $\Delta r$ and $\Delta \theta$ for to zero. The limit is called the double integral of $f$ over $R$. We define the norm $\|P\|$ of this partition of the region as $\|P\| = \max_{1 \leq k \leq n} \{\Delta r_k, \Delta \theta_k\}$. In symbols,

$$\int \int_{R} f(r, \theta) \, dA = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(r_k, \theta_k) \Delta A_k.$$
Note. To evaluate the limit above, we need to evaluate $\Delta A_k$ in terms of $\Delta r$ and $\Delta \theta$. We choose $r_k$ to be the average of the radii of the inner and outer arcs bounding the $k$th polar rectangle $\Delta A_k$. The radius of the inner arc bounding $\Delta A_k$ is then $r_k - (\Delta r/2)$. The radius of the outer arc is $r_k + (\Delta r/2)$. The area of a wedge-shaped sector of a circle having radius $r$ and angle $\theta$ is $A = \frac{1}{2} \theta r^2$, as can be seen by multiplying $\pi r^2$, the area of the circle, by $\theta/2\pi$, the fraction of the circle’s area contained in the wedge. So the areas of the circular sectors subtended by these arcs at the origin are

\[
\text{Inner radius: } \frac{1}{2} \left( r_k - \frac{\Delta r_k}{2} \right)^2 \Delta \theta
\]

\[
\text{Outer radius: } \frac{1}{2} \left( r_k + \frac{\Delta r_k}{2} \right)^2 \Delta \theta.
\]

Therefore,

\[
\Delta A_k = \text{area of large sector} - \text{area of small sector}
\]

\[
= \frac{\Delta \theta}{2} \left[ \left( r_k + \frac{\Delta r}{2} \right)^2 - \left( r_k - \frac{\Delta r}{2} \right)^2 \right]
\]

\[
= \frac{\Delta \theta}{2} (2r_k \Delta r) = r_k \Delta r \Delta \theta.
\]

Combining this result with the sum defining $S_n$ gives

\[
S_n = \sum_{k=1}^{n} f(r_k, \theta_k) r_k \Delta r \Delta \theta.
\]
As \( \|P\| \to 0 \), these sums converge to the double integral

\[
\int \int_{R} f(r, \theta) r \, dr \, d\theta.
\]

**Note.** The procedure for finding limits of integration in rectangular coordinates also works for polar coordinates. To evaluate \( \int \int_{R} f(r, \theta) \, dA \) over a region \( R \) in polar coordinates, integrating first to \( r \) and then with respect to \( \theta \) take the following steps.

1. **Sketch.** Sketch the region and label the bounding curves.

2. **Find the \( r \)-limits of integration.** Imagine a ray \( L \) from the origin cutting through \( R \) in the direction of increasing \( r \). Mark the \( r \)-values where \( L \) enters and leaves \( R \). These are the \( r \)-limits of integration. They usually depend on the angle \( \theta \) that \( L \) makes with the positive \( x \)-axis.

3. **Find the \( \theta \)-limits of integration.** Find the smallest and largest \( \theta \)-values that bound \( R \). These are the \( \theta \)-limits of integration.

**Example.** Page 876, number 6.
Definition. The \textit{area} of a closed and bounded region $R$ in the polar coordinate plane is

$$A = \iint_{R} r \, dr \, d\theta.$$ 

Example. Page 876, number 28.

Note. The procedure for changing a Cartesian integral $\iint_{R} f(x, y) \, dx \, dy$ into a polar integral has two steps. First substitute $x = r \cos \theta$ and $y = r \sin \theta$, and replace $dx \, dy$ by $r \, dr \, d\theta$ in the Cartesian integral. Then supply the polar limits of integration for the boundary $R$. The Cartesian integral then becomes

$$\iint_{R} f(x, y) \, dx \, dy = \iint_{G} f(r \cos \theta, r \sin \theta) \, r \, dr \, d\theta.$$ 

Example. Page 876, number 10.

Examples. Page 876, numbers 38, 41.