Definition. *Cylindrical coordinates* represent a point $P$ in space by ordered triples $(r, \theta, z)$ in which

1. $r$ and $\theta$ are polar coordinates for the vertical projection of $P$ on the $xy$-plane

2. $z$ is the rectangular vertical coordinate.
Note. The equations relating rectangular \((x, y, z)\) and cylindrical \((t, \theta, z)\) coordinates are

\[
x = r \cos \theta, \quad y = r \sin \theta, \quad z = z
\]

\[
r^2 = x^2 + y^2, \quad \tan \theta = y/x.
\]

Note. In cylindrical coordinates, the equation \(r = a\) describes not just a circle in the \(xy\)-plane but an entire cylinder about the \(z\)-axis. The \(z\)-axis is given by \(r = 0\). The equation \(\theta = \theta_0\) describes the plane that contains the \(z\)-axis and makes an angle \(\theta_0\) with the positive \(x\)-axis. And, just as in rectangular coordinates, the equation \(z = z_0\) describes a plane perpendicular to the \(z\)-axis.

Figure 15.43, Page 894
**Note.** When computing triple integrals over a region $D$ in cylindrical coordinates, we partition the region into $n$ small cylindrical wedges, rather than into rectangular boxes. In the $k$th cylindrical wedge, $r$, $\theta$ and $z$ change by $\Delta r_k$, $\Delta \theta_k$, and $\Delta z_k$, and the largest of these numbers among all the cylindrical wedges is called the *norm* of the partition. We define the triple integral as a limit of Riemann sums using these wedges. The volume of such a cylindrical wedge $\Delta V_k$ is obtained by taking the area $\Delta A_k$ of its base in the $r\theta$-plane and multiplying by the height $\Delta z$. For a point $(r_k, \theta_k, z_k)$ in the center of the $k$th wedge, we calculated in polar coordinates that $\Delta A_k = r_k \Delta r_k \Delta \theta_k$. So $\Delta V_k = \Delta z_k r_k \Delta r_k \Delta \theta_k$ and a Riemann sum for $f$ over $D$ has the form

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k, z_k) \Delta z_k r_k \Delta r_k \Delta \theta_k.$$  

The triple integral of a function $f$ over $D$ is obtained by taking a limit of such Riemann sums with partitions whose norms approach zero:

$$\lim_{\|P\| \to 0} S_n = \int \int \int_D f \, dV = \int \int \int_D f \, dz \, r \, dr \, d\theta.$$
Example. Page 901, number 4.

How to Integrate in Cylindrical Coordinates

To evaluate $\int \int \int_D f(r, \theta, z) \, dV$ over a region $D$ in space in cylindrical coordinates, integrating first with respect to $z$, then with respect to $r$, and finally with respect to $\theta$, take the following steps.

1. Sketch. Sketch the region $D$ along with its projection $R$ on the $xy$-
plane. Label the surfaces and curves that bound $D$ and $R$.

2. Find the $z$-limits of integration. Draw a line $M$ passing through a typical point $(r, \theta)$ of $R$ parallel to the $z$-axis. As $z$ increases, $M$ enters $D$ at $z = g_1(r, \theta)$ and leaves at $z = g_2(r, \theta)$. These are the $z$-limits of integration.
3. **Find the r-limits of integration.** Draw a ray $L$ through $(r, \theta)$ from the origin. The ray enters $R$ at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the $r$-limits of integration.

4. **Find the $\theta$-limits of integration.** As $L$ sweeps across $R$, the angle $\theta$ it makes with the positive $x$-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the $\theta$-limits if integration. The integral is

$$
\int \int \int_D f(r, \theta, z) \, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r, \theta, z) \, dz \, r \, dr \, d\theta.
$$

**Example.** Page 902, number 18.
Definition. **Spherical coordinates** represent a point $P$ in space by ordered triples $(\rho, \phi, \theta)$ in which

1. $\rho$ is the distance from $P$ to the origin (notice that $\rho > 0$).

2. $\phi$ is the angle $\vec{OP}$ makes with the positive $z$-axis ($\phi \in [0, \pi]$).

3. $\theta$ is the angle from cylindrical coordinate ($\theta \in [0, 2\pi]$).

Figure 15.47, Page 897
Note. The equation $\rho = a$ describes the sphere of radius $a$ centered at the origin. The equation $\phi = \phi_0$ describes a single cone whose vertex lies at the origin and whose axis lies along the $z$-axis.

![Diagram of spherical and cylindrical coordinates](image-url)

Figure 15.48, Page 897

Note. The equations relating spherical coordinates to Cartesian coordinates and cylindrical coordinates are

$$r = \rho \sin \theta, \quad x = r \cos \theta = \rho \sin \phi \cos \theta,$$

$$z = \rho \cos \phi, \quad y = r \sin \theta = \rho \sin \phi \sin \theta,$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.$$
Note. When computing triple integrals over a region $D$ in spherical coordinates, we partition the region into $n$ spherical wedges. The size of the $k$th spherical wedge, which contains a point $(\rho_k, \phi_k, \theta_k)$, is given by the changes $\Delta \rho_k$, $\Delta \theta_k$, and $\Delta \phi_k$ in $\rho$, $\theta$, and $\phi$. Such a spherical wedge has one edge a circular arc of length $\rho_k \Delta \phi_k$, another edge a circular arc of length $\rho_k \sin \phi_k \Delta \theta_k$, and thickness $\Delta \rho_k$. The spherical wedge closely approximates a cube of these dimensions when $\Delta \rho_k$, $\Delta \theta_k$, and $\Delta \phi_k$ are all small. It can be shown that the volume of this spherical wedge $\Delta V_k$ is $\Delta V_k = \rho_k^2 \sin \phi_k \Delta \rho_k \Delta \phi_k \Delta \theta_k$ for $(\rho_k, \phi_k, \theta_k)$ a point chosen inside the wedge. The corresponding Riemann sum for a function $f(\rho, \phi, \theta)$ is

$$S_n = \sum_{k=1}^{n} f(\rho_k, \phi_k, \theta_k) \rho_k^2 \sin \phi_k \Delta \rho_k \Delta \phi_k \Delta \theta_k.$$ 

As the norm of a partition approaches zero, and the spherical wedges get smaller, the Riemann sums have a limit when $f$ is continuous:

$$\lim_{\|P\| \to 0} S_n = \int \int \int_D f(\rho, \phi, \theta) \, dV = \int \int \int_D f(\rho, \phi, \theta) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

How to Integrate in Spherical Coordinates

To evaluate \( \iiint_D f(\rho, \phi, \theta) \, dV \) over a region \( D \) in space in spherical coordinates, integrating first with respect to \( \rho \), then with respect to \( \phi \), and finally with respect to \( \theta \), take the following steps.

1. Sketch. Sketch the region \( D \) along with its projection \( R \) on the \( xy- \)
plane. Label the surfaces and curves that bound $D$ and $R$.

2. *Find the $\rho$-limits of integration.* Draw a ray $M$ from the origin through $D$ making an angle $\phi$ with the positive $z$-axis. Also draw the projection of $M$ on the $xy$-plane (call the projection $L$). The ray $L$ makes an angle $\theta$ with the positive $x$-axis. As $\rho$ increases, $M$ enters $D$ at $\rho = g_1(\phi, \theta)$ and leaves at $\rho = g_2(\phi, \theta)$. These are the $\rho$-limits
of integration.

3. Find the $\phi$-limits of integration. For any given $\theta$, the angle $\phi$ that $M$ makes with the $z$-axis runs from $\phi = \phi_{\text{min}}$ to $\phi = \phi_{\text{max}}$. These are the $\phi$-limits of integration.

4. Find the $\theta$-limits of integration. The ray $L$ sweeps over $R$ as $\theta$ runs from $\alpha$ to $\beta$. These are the $\theta$-limits of integration. The integral is

$$\int \int \int_D f(\rho, \phi, \theta) \, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{\text{min}}}^{\phi=\phi_{\text{max}}} \int_{\rho=g_1(\phi, \theta)}^{\rho=g_2(\phi, \theta)} f(\rho, \phi, \theta) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

Example. Page 903, number 34.
Note. In summary, we have the following relationships.

\[
\begin{align*}
\text{Cylindrical to} & \quad \text{Spherical to} & \quad \text{Spherical to} \\
\text{Rectangular} & \quad \text{Rectangular} & \quad \text{Cylindrical} \\
\begin{align*}
x &= r \cos \theta & x &= \rho \sin \phi \cos \theta & r &= \rho \sin \phi \\
y &= r \sin \theta & y &= \rho \sin \phi \sin \theta & z &= \rho \cos \phi \\
z &= z & z &= \rho \cos \phi & \theta &= \theta
\end{align*}
\]

In terms of the differential of volume, we have

\[
dV = dx\,dy\,dz = dz\,r\,dr\,d\theta = \rho^2 \sin \phi \,d\rho\,d\phi\,d\theta.
\]

Examples. Page 903, number 46. Page 904, number 54.