Calculus 3 Test 2 — Spring 2011

NAME KEY STUDENT NUMBER ____________

SHOW ALL WORK!!! Partial credit will only be given for answers which are *partially correct*! Be clear and convince me that you understand what is going on. Communicate with me; write words! Use equal signs when things are equal (and don't use them when things are not equal). Be sure to include all necessary symbols, such as equal signs. If applicable, put your answer in the box provided. Each numbered problem is worth 12 points and the bonus problem is worth 12 points. **Put your calculators away!!! This is a *math* test!!!**

p698 #5

1. Find the projection of vector $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ onto vector $\mathbf{v} = 5\mathbf{j} - 3\mathbf{k}$.

Well, $\text{proj}_{\vec{v}}(\vec{u}) = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v} = \left(\frac{(1)(0)+(1)(5)+(1)(-3)}{\left(\sqrt{5^2+(-3)^2}\right)^2}\right)(5\hat{j}-3\hat{k})$

$$= \frac{2}{34}(5\hat{j}-3\hat{k}) = \frac{5}{17}\hat{j} - \frac{3}{17}\hat{k}$$

$\boxed{\frac{5}{17}\hat{j} - \frac{3}{17}\hat{k}}$

2. Find the area of the triangle with vertices $P(1, 1, 1)$, $Q(2, 1, 3)$, and $R(3, -1, 1)$.

p 705 #16

Define $\vec{PQ} = \langle 1, 0, 2\rangle$ and $\vec{PR} = \langle 2, -2, 0\rangle$. Then $|\vec{PQ}\times\vec{PR}|$ = area of parallelogram determined by $\vec{PQ}$ and $\vec{PR}$, so the area of the triangle is

$$\frac{1}{2}|\vec{PQ}\times\vec{PR}| = \frac{1}{2}\left|\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 0 & 2\\ 2 & -2 & 0\end{vmatrix}\right| = \frac{1}{2}\left|(4)\hat{i} - (-4)\hat{j} + (-2)\hat{k}\right|$$

$$= \frac{1}{2}\sqrt{(4)^2+(4)^2+(-2)^2} = \frac{1}{2}(6) = 3$$

$\boxed{3}$

$P \qquad Q \qquad R$

3. Find the equation of the plane through the points $(1, 1, -1)$, $(2, 0, 2)$, and $(0, -2, 1)$.

p 713 #23

Consider $\vec{PQ} = \langle 1, -1, 3 \rangle$ and $\vec{PR} = \langle -1, -3, 2 \rangle$. Then a normal vector for the plane is

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix} = (7)\hat{i} - (5)\hat{j} + (-4)\hat{k}.$$

So an equation is (using $P(1, 1, -1)$):

$$7(x-1) - 5(y-1) - 4(z-(-1)) = 0$$

or $7x - 5y - 4z = 6$

$$\boxed{7x - 5y - 4z = 6}$$

4. Find a parametric representation of the line tangent to the smooth curve $\mathbf{r}(t) = (\sin t)\mathbf{i} + (t^2 - \cos t)\mathbf{j} + e^t\mathbf{k}$ at $t_0 = 0$.

p 732 #19

Well, $\vec{r}'(t) = (\cos t)\hat{i} + (2t + \sin t)\hat{j} + (e^t)\hat{k}$ and $\vec{r}'(0) = \hat{i} + 0\hat{j} + \hat{k}$ is a direction vector for the line. Also, $\vec{r}(0) = 0\hat{i} - \hat{j} + \hat{k}$, so $P(0, -1, 1)$ is a point on the line. So, the line is

$$x = 0 + 1t = t$$
$$y = -1 + 0t = -1$$
$$z = 1 + 1t = 1 + t$$

$x = t$
$y = -1$
$z = 1 + t$

5. Solve the initial value problem:

p739 #15

Differential Equation: $\dfrac{d^2\mathbf{r}}{dt^2} = -32\mathbf{k}$

Initial conditions: $\mathbf{r}(0) = 100\mathbf{k}$

$\left.\dfrac{d\mathbf{r}}{dt}\right|_{t=0} = 8\mathbf{i} + 8\mathbf{j}.$

Well, $\vec{r}\,' \in \int \vec{r}\,'' \, dt = \int (-32\hat{k})\, dt$
$= (-32t)\hat{k} + \vec{C}_1$. When $t=0$,
$\vec{r}\,'(0) = (-32(0))\hat{k} + \vec{C}_1 \equiv 8\hat{i} + 8\hat{j}$
$\Rightarrow \vec{C}_1 = 8\hat{i} + 8\hat{j}$ and so
$\vec{r}\,'(t) = (-32t)\hat{k} + \vec{C}_1 = 8\hat{i} + 8\hat{j} - 32t\hat{k}$.

Next, $\vec{r}(t) \in \int \vec{r}\,'(t)\, dt = \int (8\hat{i} + 8\hat{j} - 32t\hat{k})\, dt = (8t)\hat{i} + (8t)\hat{j} - 16t^2\hat{k} + \vec{C}_2$.
When $t=0$, $\vec{r}(0) = (8\cdot 0)\hat{i} + (8\cdot 0)\hat{j} - 16(0)^2\hat{k} + \vec{C}_2 \equiv 100\hat{k}$
$\Rightarrow \vec{C}_2 = 100\hat{k}$ and so $\vec{r}(t) = (8t)\hat{i} + (8t)\hat{j} - 16t^2\hat{k} + 100\hat{k}$
$= (8t)\hat{i} + (8t)\hat{j} + (100 - 16t^2)\hat{k}$

$\boxed{(8t)\hat{i} + (8t)\hat{j} + (100-16t^2)\hat{k}}$

6. Find the length of $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + \sqrt{5}t\mathbf{k}$ for $t \in [0, \pi]$.

p745 #1

Well, $L = \int_a^b |\vec{r}\,'(t)|\, dt = \int_0^\pi |(-2\sin t)\hat{i} + (2\cos t)\hat{j} + \sqrt{5}\hat{k}|\, dt$
$= \int_0^\pi \{(-2\sin t)^2 + (2\cos t)^2 + (\sqrt{5})^2\}^{1/2} dt = \int_0^\pi \sqrt{9}\, dt = 3\pi$

$\boxed{3\pi}$

7. Find curvature κ for $\mathbf{r} = (3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 4t\mathbf{k}$. HINT: $\mathbf{T} = \dfrac{d\mathbf{r}/dt}{|d\mathbf{r}/dt|} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$ and $\kappa = \dfrac{1}{|\mathbf{v}|}\left|\dfrac{d\mathbf{T}}{dt}\right|$.

$\boxed{\begin{matrix} p751 \\ \#9 \end{matrix}}$

well, $\dfrac{d\vec{r}}{dt} = (3\cos t)\hat{i} + (-3\sin t)\hat{j} + 4\hat{k}$ and $\left|\dfrac{d\vec{r}}{dt}\right| = \sqrt{25} = 5 = |\vec{v}|$.

So $\vec{T} = \left(\frac{3}{5}\cos t\right)\hat{i} - \left(\frac{3}{5}\sin t\right)\hat{j} + \frac{4}{5}\hat{k}$, and

$\dfrac{d\vec{T}}{dt} = \left(\frac{-3}{5}\sin t\right)\hat{i} - \left(\frac{3}{5}\cos t\right)\hat{j} + 0\hat{k}$ and $\left|\dfrac{d\vec{T}}{dt}\right| = \frac{3}{5}$.

So, $\kappa = \left(\frac{1}{5}\right)\left(\frac{3}{5}\right) = \frac{3}{25}$

$$\boxed{\kappa = 3/25}$$

8. Consider $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$. One can show that $\mathbf{N}(0) = -\mathbf{i}$. Find the binormal unit vector $\mathbf{B}$ at $t = 0$. HINT: $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.

$\boxed{\begin{matrix} p756 \\ \#8 \end{matrix}}$

well, $\dfrac{d\vec{r}}{dt} = (-\sin t)\hat{i} + (\cos t)\hat{j} + \hat{k}$ and $\left|\dfrac{d\vec{r}}{dt}\right| = \sqrt{2}$.

So, $\vec{T} = \dfrac{d\vec{r}/dt}{|d\vec{r}/dt|} = \left(-\frac{1}{\sqrt{2}}\sin t\right)\hat{i} + \left(\frac{1}{\sqrt{2}}\cos t\right)\hat{j} + \left(\frac{1}{\sqrt{2}}\right)\hat{k}$

Hence, when $t=0$,

$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{-1}{\sqrt{2}}\sin 0 & \frac{1}{\sqrt{2}}\cos 0 & \frac{1}{\sqrt{2}} \\ -1 & 0 & 0 \end{vmatrix} = 0\hat{i} - \left(\frac{1}{\sqrt{2}}\right)\hat{j} + \left(\frac{1}{\sqrt{2}}\cos 0\right)\hat{k}$$

$$= \frac{-1}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$$

$$\boxed{\frac{-1}{\sqrt{2}}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}}$$

Bonus. Prove one of the following:

p.727 (a) Let $\mathbf{r}(t) = \mathbf{f}(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function and let $\mathbf{L}$ be a vector. State the definition of $\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{L}$.

p.730 (b) Use the definition of derivative to prove that

$$\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t).$$

(a) Let $\vec{r}(t) = f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k}$ be a vector function defined on an open interval containing t_0 except possibly at t_0 itself, and let $\vec{L}$ be a vector. We say that $\vec{r}$ has <u>limit</u> $\vec{L}$ as t approaches t_0, denoted $\lim_{t\to t_0} \vec{r}(t) = \vec{L}$, if for every number $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that $|\vec{r}(t) - \vec{L}| < \varepsilon$ whenever $0 < |t - t_0| < \delta$.

(b) $\frac{d}{dt}[\vec{u}(t)\cdot\vec{v}(t)] = \frac{d}{dt}[u_1(t)v_1(t) + u_2(t)v_2(t) + u_3(t)v_3(t)]$

where $\vec{u}(t) = u_1(t)\,\hat{i} + u_2(t)\,\hat{j} + u_3(t)\,\hat{k}$

and $\vec{v}(t) = v_1(t)\,\hat{i} + v_2(t)\,\hat{j} + v_3(t)\,\hat{k}$

$= \frac{d}{dt}[u_1(t)v_1(t)] + \frac{d}{dt}[u_2(t)v_2(t)] + \frac{d}{dt}[u_3(t)v_3(t)]$

$= \left(u_1'(t)v_1(t) + u_1(t)v_1'(t)\right) + \left(u_2'(t)v_2(t) + u_2(t)v_2'(t)\right)$

$+ \left(u_3'(t)v_3(t) + u_3(t)v_3'(t)\right)$ by the Product Rule

$= u_1'(t)v_1(t) + u_2'(t)v_2(t) + u_3'(t)v_3(t)$

$+ u_1(t)v_1'(t) + u_2(t)v_2'(t) + u_3(t)v_3'(t)$

$= \vec{u}'\cdot\vec{v} + \vec{u}\cdot\vec{v}'.$