# Calculus 3 Test 3 — Spring 2011

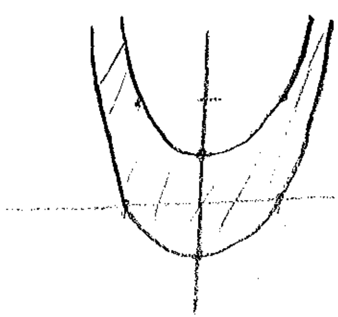
NAME KEY STUDENT NUMBER

**SHOW ALL WORK!!!** Partial credit will only be given for answers which are *partially correct*! Be clear and convince me that you understand what is going on. Communicate with me; write words! Use equal signs when things are equal (and don't use them when things are not equal). Be sure to include all necessary symbols, such as the partial symbol. If applicable, put your answer in the box provided. Each numbered problem is worth 12 points and the bonus problem is worth 12 points. **Put your calculators away!!! This is a** ***math*** **test!!!**

**1.** Find and sketch the domain of $f(x, y) = \cos^{-1}(y - x^2)$. Is the domain open, closed or neither? Is the domain bounded or unbounded?

$\boxed{\text{p 271 \#9}}$

We need $-1 \le y - x^2 \le 1$ or $x^2 - 1 \le y \le x^2 + 1$. So the domain is $\{(x,y) \mid x^2 - 1 \le y \le x^2 + 1\}$.



Domain is closed (contains its boundary points) and unbounded.

**2.** (a) State "Alternate Definition 2" for $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$.

Let $(x_0, y_0)$ be a limit point of the domain of $f$. We say that a function $f(x,y)$ approaches the limit $L$ as $(x,y)$ approaches $(x_0, y_0)$, if for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $(x,y)$ in the domain of $f$,

$$|f(x,y) - L| < \varepsilon \text{ whenever } 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta.$$

p780 #19

(b) Evaluate $\lim_{(x,y)\to(2,0)} \frac{\sqrt{2x-y}-2}{2x-y-4}$. $= \lim_{(x,y)\to(2,0)} \frac{(\sqrt{2x-y}-2)}{(\sqrt{2x-y}-2)(\sqrt{2x-y}+2)}$

$= \lim_{(x,y)\to(2,0)} \left(\frac{1}{\sqrt{2x-y}+2}\right) = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$

$\boxed{1/4}$

3. Find all second-order partial derivatives of $h(x,y) = xe^y + y + 1$.

p791 #44

$h_x = e^y \qquad h_y = xe^y + 1$

$\boxed{h_{xx} = 0 \qquad h_{yy} = xe^y \qquad h_{xy} = e^y}$

4. The three-dimensional Laplace equation is $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$. It describes the temperature distribution in a room at equilibrium (among other physical situations). Show that $f(x,y,z) = 2z^3 - 3(x^2+y^2)z$ satisfies the three-dimensional Laplace equation.

p792 #74

Well, $f_x = -6xz$, $f_y = -6yz$, $f_z = 6z^2 - 3(x^2+y^2)$

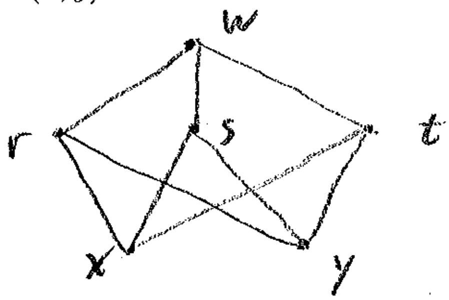
and $f_{xx} = -6z$, $f_{yy} = -6z$, $f_{zz} = 12z$.

So $f_{xx} + f_{yy} + f_{zz} = (-6z) + (-6z) + (12z) = 0.$ ✓

**5.** Use a branch diagram and the Chain Rule to find $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ for $w = f(r, s, t)$, $r = g(x, y)$, $s = h(x, y)$, $t = k(x, y)$.

p800 #16

We have:




So

$$\boxed{\begin{aligned} \frac{\partial w}{\partial x} &= \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x} \\ \frac{\partial w}{\partial y} &= \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y} \end{aligned}},$$

**6.** Find the directional derivative of $g(x, y, z) = 3e^x \cos(yz)$ in the direction $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ at the point $P_0(0, 0, 0)$. (Notice that $\mathbf{v}$ is not a unit vector.)

p 808 #17

First, let $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$.

Next, $\nabla g = (3e^x\cos(yz))\hat{i} + (3e^x(-\sin(yz))[z])\hat{j}$
$+ (3e^x(-\sin(yz))[y])\hat{k}$

and $\nabla g|_{(0,0,0)} = 3\hat{i} + 0\hat{j} + 0\hat{k}$.

So $D_{\vec{u}, P_0}[g] = \nabla g \cdot \vec{u}$
$= (3\hat{i} + 0\hat{j} + 0\hat{k}) \cdot (\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}) = \boxed{2}$

7. Find the equation of the plane tangent to $\cos(\pi x) - x^2y + e^{xz} + yz = 4$ at the point $P_0(0,1,2)$.

p817 #5a

Let $f(x,y) = \cos(\pi x) - x^2y + e^{xz} + yz$. We need

$$\nabla f = \left(-\sin(\pi x)\pi - 2xy + e^{xz}[z]\right)\hat{i} + \left(-x^2 + z\right)\hat{j} + \left(e^{xz}[x] + y\right)\hat{k}.$$

$\nabla f\big|_{(0,1,2)} = 2\hat{i} + 2\hat{j} + \hat{k}$. So plane is

$$2(x-0) + 2(y-1) + 1(z-2) = 0$$

$$\boxed{2x + 2(y-1) + (z-2) = 0}$$

8. Find all local maxima, local minima, and saddle points for $f(x,y) = x^3 + 3xy + y^3$. HINT: The Hessian of $f(x,y)$ is $f_{xx}f_{yy} - (f_{xy})^2$.

p827 #14

Well, $\nabla f = (3x^2 + 3y)\hat{i} + (3x + 3y^2)\hat{j}$

Set $\nabla f = \vec{0} \Rightarrow \begin{cases} 3x^2 + 3y = 0 \Rightarrow y = -x^2 \\ 3x + 3y^2 = 0 \Rightarrow x + (-x^2)^2 = 0 \Rightarrow x(1 + x^3) = 0 \end{cases}$

$\Rightarrow x = 0, y = 0$ and $x = -1, y = -1$

So critical points are $(0,0)$ and $(-1,-1)$.

Next, $f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = 3$. So

Hessian $= (6x)(6y) - (3)^2 = 36xy - 9$.

At $(0,0)$, Hessian $= -9 \Rightarrow$ saddle point.

At $(-1,-1)$, Hessian $= 27 > 0$, and $f_{xx} = -6 \Rightarrow$ local MAX.

Local MAX at $(-1,-1)$ & Saddle point at $(0,0)$

**Bonus.** Find the equation(s) of the normal line to $\cos(\pi x) - x^2y + e^{xz} + yz = 4$ at the point $P_0(0,1,2)$. (This is the same surface considered in number 7.)

p 817
#5b

Since $\nabla f|_{(0,1,2)} = 2\vec{i} + 2\vec{j} + \vec{k}$, the normal line is

$$x = x_0 + (f_x)t = 0 + 2t$$
$$y = y_0 + (f_y)t = 1 + 2t$$
$$z = z_0 + (f_z)t = 2 + t$$

at $P_0$ ↑

$x = 2t$ $\quad z = 2 + t$
$y = 1 + 2t$