Calculus 3 Test 4 — Spring 2011

NAME KEY STUDENT NUMBER ____________

SHOW ALL WORK!!! Partial credit will only be given for answers which are *partially correct*! Be clear and convince me that you understand what is going on. Communicate with me; write words! Use equal signs when things are equal (and don't use them when things are not equal). Be sure to include all necessary symbols, such as the partial symbol and multiple integral symbols. If applicable, put your answer in the box provided. Each numbered problem is worth 12 points and the bonus problem is worth 12 points. **Put your calculators away!!! This is a *math* test!!!**

1. Consider the function $f(x, y)$ defined on the rectangular region $R = \{(x, y) \mid x \in [a, b], y \in [c, d]\}$. Define the double integral of f over R. Be sure to include all necessary parts of the definition, which include: partition, norm of the partition, Riemann sum, and double integral.

15.1 notes, pp. 1-3

Subdivide R into small rectangles using lines parallel to the axes. These are a partition of R. Number the rectangles 1 to n, let Δx_k be the width, Δy_k the height, and $\Delta A_k = \Delta x_k \Delta y_k$ the area of rectangle k. The norm of the partition is $\|P\| = \max_{1 \le k \le n} \{\Delta x_k, \Delta y_k\}$. Choose point (x_k, y_k) in rectangle k and define the Riemann sum $S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$.

The double integral is

$$\iint_R f(x,y)\,dA = \lim_{\|P\| \to 0} \left(\sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k \right).$$

2. Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the square $R : x \in [-1, 1], y \in [-1, 1]$.

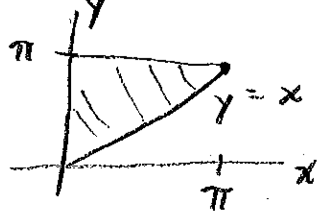
p859 #23

$$V = \iint_R f(x,y)\,dA = \int_{x=-1}^{x=1} \int_{y=-1}^{y=1} (x^2+y^2)\,dy\,dx = \int_{x=-1}^{x=1} \left(yx^2 + \tfrac{1}{3}y^3\right)\Big|_{y=-1}^{y=1} dx$$

$$= \int_{-1}^{1} \left(\left(x^2 + \tfrac{1}{3}\right) - \left(-x^2 - \tfrac{1}{3}\right)\right) dx = \int_{-1}^{1} \left(2x^2 + \tfrac{2}{3}\right) dx = \left(\tfrac{2}{3}x^3 + \tfrac{2}{3}x\right)\Big|_{-1}^{1}$$

$$= \boxed{\frac{8}{3}}$$

Area is:



So x ranges from 0 to y, and y ranges from 0 to π.

p 866 #47

3. Evaluate $\int_0^\pi \int_x^\pi \frac{\sin y}{y}\,dy\,dx$.

$$= \int_{y=0}^{y=\pi} \int_{x=0}^{x=y} \frac{\sin y}{y}\,dx\,dy = \int_{y=0}^{y=\pi} \left(x\,\frac{\sin y}{y}\right)\Big|_{x=0}^{x=y} dy = \int_0^\pi (\sin y)\,dy$$

$$= -\cos y\Big|_0^\pi = -\cos(\pi) + \cos(0) = -(-1) + (1) = 2$$

$$\boxed{2}$$

4. Find the average value of $f(x,y) = \sin(x+y)$ over the rectangle $x \in [0,\pi], y \in [0,\pi]$.

p870 #19a

$$(\text{av. value}) = \frac{1}{\text{area of } R} \iint_R f(x,y)\,dA = \frac{1}{\pi^2} \int_{x=0}^{x=\pi} \int_{y=0}^{y=\pi} \sin(x+y)\,dy\,dx$$

$$= \frac{1}{\pi^2} \int_{x=0}^{x=\pi} \left(-\cos(x+y)\Big|_{y=0}^{y=\pi}\right) dy = \frac{1}{\pi^2} \int_{x=0}^{x=\pi} -\cos(x+\pi) + \cos(x)\,dx$$

$$= \frac{1}{\pi^2}\left(-\sin(x+\pi) + \sin(x)\right)\Big|_0^\pi = \frac{1}{\pi^2}\Big(\big(-\sin(2\pi) + \sin(\pi)\big) - \big(-\sin(\pi) + \sin 0\big)\Big)$$

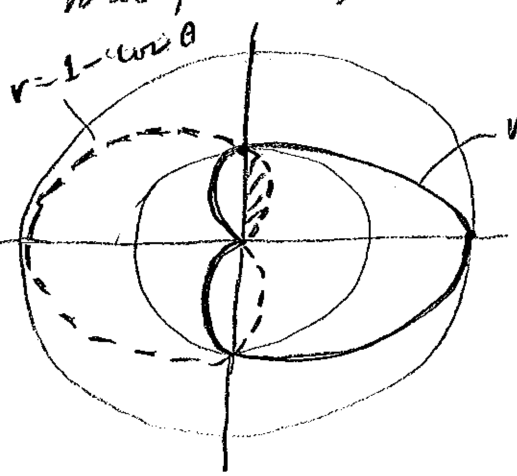
$$= \frac{1}{\pi^2}(0) = 0$$

$$\boxed{0}$$

5. **Set up a double integral for** the area of the region common to the interiors of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$. HINT: The differential of area in polar coordinates is $dA = r\,dr\,d\theta$.

p 876 #32

Well, the region is



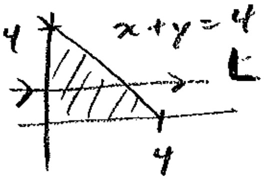
By symmetry, we take 4 times the area in the first quadrant:

$$A = 4 \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=1-\cos\theta} 1\,r\,dr\,d\theta.$$

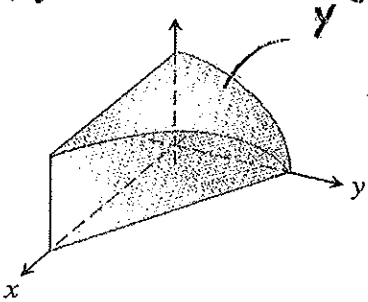
p 885 #31

6. Set up a triple integral for the volume of the region in the first octant bounded by the coordinate planes, the plane $x + y = 4$, and the cylinder $y^2 + 4z^2 = 16$.

Well, z ranges from $z = 0$ to $z = \sqrt{4 - y^2/4}$. In the xy-plane:



$y^2 + 4z^2 = 16$ or $z = +\sqrt{4 - y^2/4}$



y ranges from $y = 0$ to $y = 4 - x$. x ranges from $x = 0$ to $x = 4$.

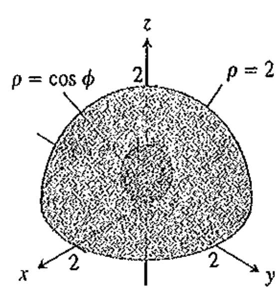
So,

$$V = \int_{x=0}^{x=4} \int_{y=0}^{y=4-x} \int_{z=0}^{z=\sqrt{4-y^2/4}} 1 \, dz \, dy \, dx$$

7. Set up a triple integral in spherical coordinates for the volume of the region between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2$, for $z \geq 0$. HINT: The differential of volume in spherical coordinates is $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$.

p 903 #33

Well, ρ ranges from $\rho = \cos\phi$ to $\rho = 2$; ϕ ranges from $\phi = 0$ to $\phi = \pi/2$; θ ranges from $\theta = 0$ to $\theta = 2\pi$. So

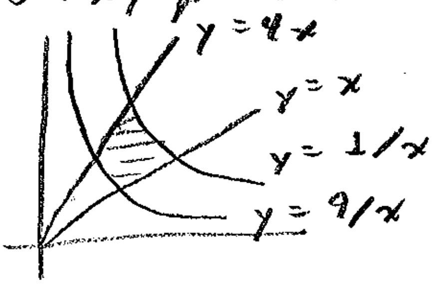


$$V = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi/2} \int_{\rho=\cos\phi}^{\rho=2} 1 \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

p 913 #9

8. Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1$, $xy = 9$ and the lines $y = x$, $y = 4x$. Use the transformation $x = u/v$, $y = uv$ with $u > 0$ and $v > 0$ to rewrite $\int\int_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\, dy$ as an integral over an appropriate region G in the uv-plane. HINT: The Jacobian is $J(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$.

In xy-plane R is:

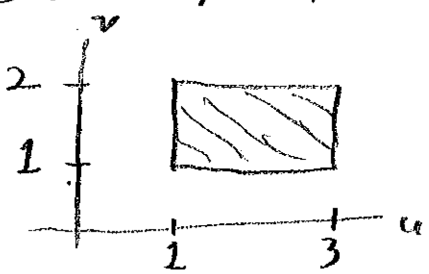


Next, $J(u,v) = \begin{vmatrix} 1/v & -u/v^2 \\ v & u \end{vmatrix} = \frac{u}{v} - \frac{-u}{v} = \frac{2u}{v}$.

So, integral becomes

$$\int_{u=1}^{u=3} \int_{v=1}^{v=2} \left(\sqrt{\frac{uv}{u/v}} + \sqrt{\frac{u}{v} \cdot uv}\right) \left|\frac{2u}{v}\right| dv\, du$$

In uv-plane, R is:



Bonus. The usual way to evaluate the improper integral $I = \int_0^\infty e^{-x^2}\, dx$ is first to calculate its square:

p 876 # 41 a

$$I^2 = \left(\int_0^\infty e^{-x^2}\, dx\right)\left(\int_0^\infty e^{-y^2}\, dy\right) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy.$$

Evaluate the last integral using polar coordinates and solve the resulting equation for I.

In polar coordinates, $r^2 = x^2 + y^2$ and $dx\,dy = r\,dr\,d\theta$.
The first quadrant is: r ranges from $r = 0$ to $r = \infty$; θ ranges from $\theta = 0$ to $\theta = \pi/2$. So $I^2 = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=\infty} e^{-r^2}\, r\, dr\, d\theta$

$$= \int_{\theta=0}^{\theta=\pi/2} \lim_{b\to\infty} \left(-\frac{1}{2} e^{-r^2}\right)\Big|_0^b d\theta = \int_{\theta=0}^{\theta=\pi/2} \lim_{b\to\infty} \left(-\frac{1}{2} e^{-b^2} + \frac{1}{2} e^0\right) d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\, d\theta = \frac{\pi}{4} \Rightarrow I = \frac{\sqrt{\pi}}{2}$$

$\boxed{\sqrt{\pi}/2}$