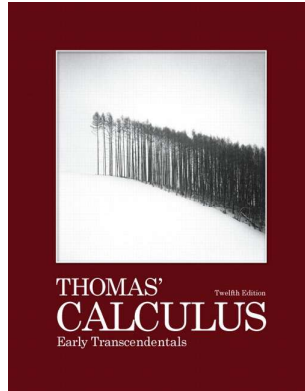


Calculus 3

Chapter 15. Multiple Integrals

15.1. Double and Iterated Integrals over Rectangles—Examples and Proofs of Theorems



Exercise 15.1.6

Exercise 15.1.6. Evaluate the double integral $\int_0^3 \int_{-2}^0 (x^2y - 2xy) dy dx$.

Solution. We evaluate this iterated integral by first integrating with respect to y and then with respect to x . So, by the Fundamental Theorem of Calculus (Part 2) we have:

$$\begin{aligned} \int_0^3 \int_{-2}^0 (x^2y - 2xy) dy dx &= \int_0^3 \left(x^2 \frac{y^2}{2} - 2x \frac{y^2}{2} \right) \Big|_{y=-2}^{y=0} dx \\ &= \int_0^3 (x^2 y^2 / 2 - xy^2) \Big|_{y=-2}^{y=0} dx = \int_0^3 0 - (x^2(-2)^2 / 2 - x(-2)^2) dx \\ &= \int_0^3 -2x^2 + 4x dx = \left(-2 \frac{x^3}{3} + 4 \frac{x^2}{2} \right) \Big|_{x=0}^{x=3} \\ &= (-2(3)^3 / 3 + 4(3)^2 / 2) - (0) = -18 + 18 = 0. \end{aligned}$$

□

Exercise 15.1.16

Exercise 15.1.16

Exercise 15.1.16. Evaluate the double integral over the region R :

$$\iint_R y \sin(x+y) dA \text{ where } R = \{(x, y) \mid -\pi \leq x \leq 0, 0 \leq y \leq \pi\}.$$

Solution. By Fubini's Theorem, First Form (Theorem 1), the integral can be written as an iterated integrals as follows:

$$\iint_R y \sin(x+y) dA = \int_{-\pi}^0 \int_0^\pi y \sin(x+y) dy dx = \int_0^\pi \int_{-\pi}^0 y \sin(x+y) dx dy.$$

We choose to integrate with respect to x first:

$$\begin{aligned} \iint_R y \sin(x+y) dA &= \int_0^\pi \int_{-\pi}^0 y \sin(x+y) dx dy \\ &= \int_0^\pi -y \cos(x+y) \Big|_{x=-\pi}^{x=0} dy = \int_0^\pi -y \cos(0+y) - (-y \cos(\pi+y)) dy \end{aligned}$$

Exercise 15.1.16

Exercise 15.1.16 (continued)

Solution (continued).

$$\begin{aligned} &= \int_0^\pi -y \cos y + y(-\cos y) \text{ since } \cos(\pi+y) = -\cos(-y) \text{ for all } y \in \mathbb{R} \\ &= \int_0^\pi -2y \cos y dy \text{ now let } u = y \text{ and } dv = \cos y dy, \\ &\quad \text{so that } du = dy \text{ and } v = \sin y \\ &= -2 \left(y \sin y - \int \sin y dy \right) \Big|_{y=0}^{y=\pi} \text{ by integration by parts} \\ &= -2 (y \sin y - (-\cos y)) \Big|_{y=0}^{y=\pi} = -2(\pi \sin \pi + \cos \pi) + 2(0 + \cos 0) \\ &= -2(0 + (-1)) + 2(1) = 4. \end{aligned}$$

□

Exercise 15.1.28

Exercise 15.1.28. Find the volume of the region bounded above by the surface $z = 4 - y^2$ and below by the rectangle $R = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 2\}$.

Solution. First, notice that $z = f(x, y) = 4 - y^2$ is nonnegative over R since $0 \leq y \leq 2$ for $(x, y) \in R$. So by the definition of volume and by Fubini's Theorem, First Form (Theorem 1), we have

$$V = \iint_R f(x, y) \, dA = \int_0^1 \int_0^2 (4 - y^2) \, dy \, dx = \int_0^1 \int_0^2 (4 - y^2) \, dx \, dy.$$

We choose to integrate with respect to x first:

$$\begin{aligned} V &= \int_0^2 \int_0^1 (4 - y^2) \, dx \, dy = \int_0^2 (4x - xy^2) \Big|_{x=0}^{x=1} \, dy = \int_0^2 (4 - y^2) \, dy \\ &= \left(4y - \frac{y^3}{3}\right) \Big|_{y=0}^{y=2} = (4(2) - (2^3)/3) - (0) = 8 - 8/3 = 16/3. \end{aligned}$$

□