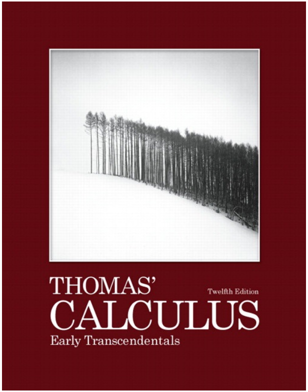
Calculus 3

Chapter 15. Multiple Integrals

15.1. Double and Iterated Integrals over Rectangles—Examples and Proofs of Theorems



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Solution. We evaluate this iterated integral by first integrating with respect to y and then with respect to x. So, by the Fundamental Theorem of Calculus (Part 2) we have:

$$\int_0^3 \int_{-2}^0 (x^2y - 2xy)\, dy\, dx = \int_0^3 \left(x^2\frac{y^2}{2} - 2x\frac{y^2}{2} \right) \Bigg|_{y=-2}^{y=0} dx$$

$$= \int_0^3 (x^2y^2/2 - xy^2)|_{y=-2}^{y=0}\, dx = \int_0^3 0 - (x^2(-2)^2/2 - x(-2)^2)\, dx$$

$$= \int_0^3 -2x^2 + 4x\, dx = \left(-2\frac{x^3}{3} + 4\frac{x^2}{2} \right) \Bigg|_{x=0}^{x=3}$$

$$= (-2(3)^3/3 + 4(3)^2/2) - (0) = -18 + 18 = 0.$$

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$$\iint_R y \sin(x+y)\, dA \text{ where } R = \{(x,y) \mid -\pi \leq x \leq 0, 0 \leq y \leq \pi\}.$$

Solution. By Fubini's Theorem, First Form (Theorem 1), the integral can be written as an iterated integrals as follows:

$$\iint_R y \sin(x+y)\, dA = \int_{-\pi}^{0} \int_{0}^{\pi} y \sin(x+y)\, dy\, dx = \int_{0}^{\pi} \int_{-\pi}^{0} y \sin(x+y)\, dx\, dy.$$

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We choose to integrate with respect to x first:

$$\iint_R y\sin(x+y)\,dA = \int_{0}^{\pi}\int_{-\pi}^{0} y\sin(x+y)\,dx\,dy.$$

$$= \int_{0}^{\pi} -y\cos(x+y)\big|_{x=-\pi}^{x=0}\,dy = \int_{0}^{\pi} -y\cos(0+y) - (-y\cos(\pi+y))\,dy$$

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Exercise 15.1.16 (continued)

Solution (continued).

$$\begin{aligned}
&= \int_0^{\pi} -y\cos y + y(-\cos y) \text{ since } \cos(\pi + y) = -\cos(-y) \text{ for all } y \in \mathbb{R} \\
&= \int_0^{\pi} -2y\cos y\, dy \text{ now let } u = y \text{ and } dv = \cos y\, dy, \\
&\quad \text{so that } du = dy \text{ and } v = \sin y \\
&= -2\left(y\sin y - \int \sin y dy \right)\Bigg|_{y=0}^{y=\pi} \text{ by integration by parts} \\
&= -2\left(y\sin y - (-\cos y)\right)\Big|_{y=0}^{y=\pi} = -2(\pi\sin\pi + \cos\pi) + 2(0 + \cos 0) \\
&= -2(0 + (-1)) + 2(1) = 4.
\end{aligned}$$

□

Exercise 15.1.28

Exercise 15.1.28. Find the volume of the region bounded above by the surface $z = 4 - y^2$ and below by the rectangle $R = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 2\}$.

Solution. First, notice that $z = f(x, y) = 2 - y^2$ is nonnegative over R since $0 \leq y \leq 2$ for $(x, y) \in R$. So by the definition of volume and by Fubini's Theorem, First Form (Theorem 1), we have

$$V = \iint_R f(x, y)\, dA = \int_0^1 \int_0^2 4 - y^2\, dy\, dx = \int_0^2 \int_0^1 4 - y^2\, dx\, dy.$$

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We choose to integrate with respect to x first:

$$V = \int_0^2 \int_0^1 4 - y^2\, dx\, dy = \int_0^2 (4x - xy^2)\Big|_{x=0}^{x=1} dy = \int_0^2 (4 - y^2) - (0)\, dy$$

$$= \left(4y - \frac{y^3}{3}\right)\Big|_{y=0}^{y=2} = (4(2) - (2^3)/3) - (0) = 8 - 8/3 = 16/3.$$

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