Calculus 3

Chapter 15. Multiple Integrals

15.4. Double Integrals in Polar Form—Examples and Proofs of Theorems

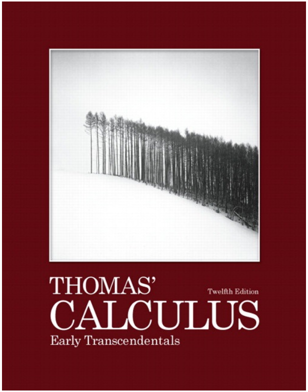

Table of contents

Exercise 15.4.6

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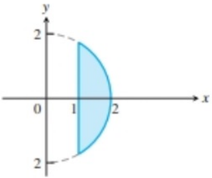

Solution. Since $x = r\cos\theta$ then along the vertical line $x = 1$ we have $1 = r\cos\theta$ or $r = 1/\cos\theta = \sec\theta$. Along the circle we have $r = 2$. So we can describe the region in terms of r-limits. A typical ray from the origin enters the region where $r = \sec\theta$ and leaves where $r = 2$. To find θ-limits, notice the following triangle:

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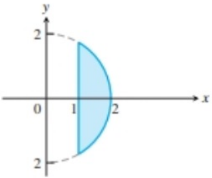

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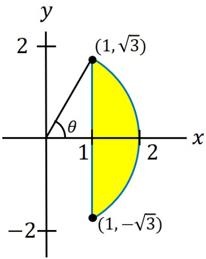

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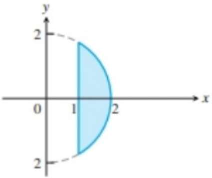

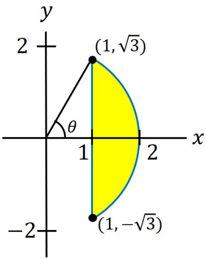

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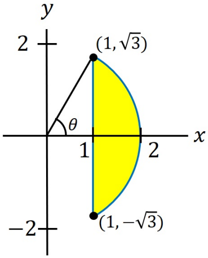

Solution (continued). We have $\cos\theta = 1/2$ so that $\theta = \cos^{-1}(1/2) = \pi/3$. So the upper θ-limit is $\pi/3$ and, by symmetry, the lower θ-limit is $-\pi/3$. In particular, an integral of $f(r\,\theta)$ over the region is of the form

$$\int_{-\pi/3}^{\pi/3}\int_{\sec\theta}^{2} f(r,\theta)\,dr\,d\theta.$$

□

Exercise 15.4.10

Exercise 15.4.10. Change the integral into an equivalent polar integral. Then evaluate the polar integral:

$$\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2)\, dx\, dy.$$

Solution. With $x = \sqrt{1 - y^2}$ we have $x^2 = 1 - y^2$ where $x \geq 0$ and $x^2 + y^2 = 1$ $x \geq 0$. This is the upper half of the unit circle centered at the origin. With y ranging from 0 to 1 we then have the region:

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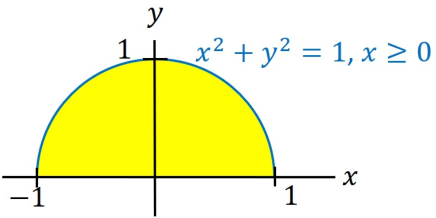

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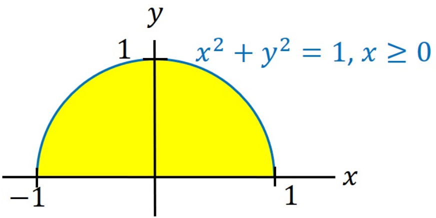

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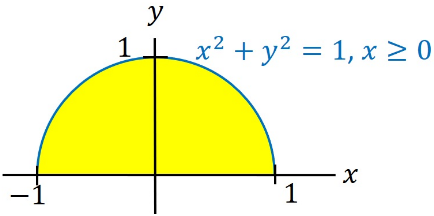

Solution (continued). We can take the r-limits as 0 to 1 and the θ-limits as 0 to π. Since $r^2 = x^2 + y^2$, the integral becomes

$$\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2 + y^2)\,dx\,dy = \int_0^{\pi} \int_0^1 r^2 r\,dr\,d\theta = \int_0^{\pi} \frac{1}{4} r^4 \Big|_{r=0}^{r=1} d\theta$$

$$= \int_0^{\pi} \frac{1}{4}\,d\theta = \frac{1}{4}\theta\Big|_0^{\pi} = \frac{\pi}{4}.$$

□

Exercise 15.4.28

Exercise 15.4.28. Find the area of the region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

Solution. The graphs of the cardioid (which is Exercise #1 on page 652 in Section 11.4) and the circle are:

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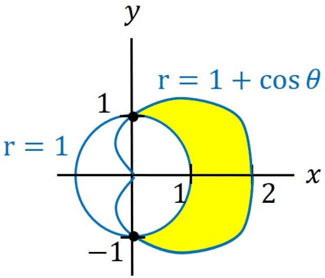

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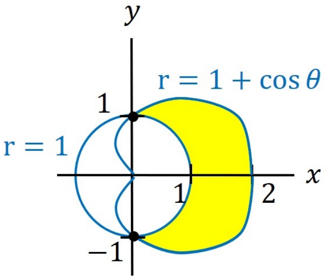

So the r-limits of the region are 1 and $1 + \cos\theta$ and the θ-limits are $-\pi/2$ and $\pi/2$.

Exercise 15.4.28 (continued)

Solution (continued). So the area of the region is

$$\begin{aligned}
A &= \iint_R r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta} r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left(\frac{1}{2}r^2\Big|_{r=1}^{r=1+\cos\theta}\right)d\theta \\
&= \int_{-\pi/2}^{\pi/2}\left(\frac{1}{2}(1+\cos\theta)^2 - \frac{1}{2}\right)d\theta = \int_{-\pi/2}^{\pi/2}\left(\cos\theta + \frac{1}{2}\cos^2\theta\right)d\theta \\
&= \sin\theta + \frac{1}{2}\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_{\theta=-\pi/2}^{\theta=\pi/2} \text{since } \int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C, \\
&\quad \text{by Example 9 in Section 5.5} \\
&= \left(\sin(\pi/2) + \frac{1}{2}\left(\frac{(\pi/2)}{2} + \frac{\sin 2(\pi/2)}{4}\right)\right) \\
&\quad - \left(\sin(-\pi/2) + \frac{1}{2}\left(\frac{(-\pi/2)}{2} + \frac{\sin 2(\pi/2)}{4}\right)\right) \\
&= (1 + (1/2)(\pi/4 + 0)) - (-1 + (1/2)(-\pi/4 + 0)) = 2 + \pi/4. \quad \square
\end{aligned}$$

Exercise 15.4.38

Exercise 15.4.38. *Converting to a Polar Integral.* Integrate $f(x,y) = \dfrac{\ln(x^2+y^2)}{x^2+y^2}$ over the region $1 \le x^2+y^2 \le e^2$.

Solution. We have $r^2 = x^2+y^2$ so $\dfrac{\ln(x^2+y^2)}{x^2+y^2} = \dfrac{\ln r}{r}$. The region is an annulus with inner radius 1 and outer radius e^2:

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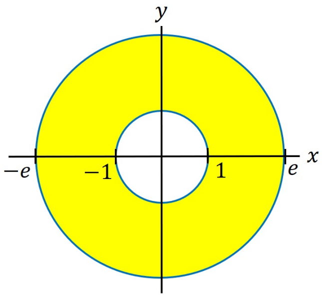

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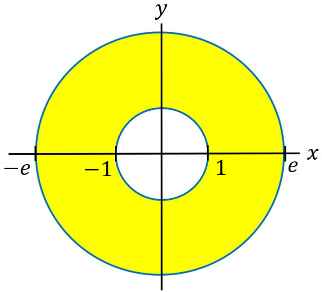

Exercise 15.4.38 (continued)

Solution. We describe this with r-limits of 1 and e^2, and θ-limits of 0 and 2π. Since $r^2 = x^2 + y^2$ and $dx\,dy = r\,dr\,d\theta$, the integral is then:

$$\iint_R \frac{\ln(x^2+y^2)}{x^2+y^2}\,dx\,dy = \int_0^{2\pi}\int_1^{e^2} \frac{\ln r}{r} r\,dr\,d\theta = \int_0^{2\pi}\int_1^{e^2} \ln r\,dr\,d\theta$$

$$= \int_0^{2\pi} (f\ln r - r)\Big|_{r=1}^{r=e^2} d\theta \text{ since } \int \ln x\,dx = x\ln x - x + C$$

by Example 2 in Section 8.1 on page 456

$$= \int_0^{2\pi} ((e^2\ln e^2 - e^2)(1\ln 1 - 1))\,d\theta = \int_0^{2\pi} ((2e^2 - e^2) - (0-1))\,d\theta$$

$$= \int_0^{2\pi} (e^2+1)\,d\theta = (e^2+1)\theta\Big|_0^{2\pi} = 2\pi(e^2+1). \quad \square$$

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Exercise 15.4.41

Exercise 15.4.41. *Converting to Polar Integrals.*

(a) The usual way to evaluate the improper integral $I = \int_0^\infty e^{-x^2}\,dx$ is first to calculate its square:

$$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-x^2}\,dx\right) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy.$$

Evaluate the last integral using polar coordinates and solve the resulting equation. for I.

Solution. The double integral is over the first quadrant of the Cartesian plane. So in polar coordinated we have the r-limits of 0 and ∞ and the θ-limits of 0 and $\pi/2$. Since $r^2 = x^2 + y^2$ and $dx\,dy = r\,dr\,d\theta$ then we have: ...

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Exercise 15.4.41 (continued)

Solution (continued). ...

$$\begin{aligned} I^2 &= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2} r\,dr\,d\theta \\ &= \int_0^{\pi/2}\left(\lim_{b\to\infty}\int_0^b re^{-r^2}\,dr\right)\,d\theta = \int_0^{\pi 2}\left(\lim_{b\to\infty}\left(\frac{-e^{-r^2}}{2}\right)\Bigg|_{r=0}^{r=b}\right)\,d\theta \\ &= \int_0^{\pi/2}\lim_{b\to\infty}\left(\frac{-e^{-b^2}}{2} - \frac{-e^{-(0)^2}}{2}\right)\,d\theta \\ &= \int_0^{\pi/2}(0+1/2)\,d\theta = (\theta/2)\Big|_{\theta=0}^{\theta=\pi/2} = \pi/4. \end{aligned}$$

So $I = \int_0^\infty e^{-x^2}\,dx = \sqrt{\pi/4} = \sqrt{\pi}/2$. □

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So $I = \int_0^\infty e^{-x^2}\,dx = \sqrt{\pi/4} = \sqrt{\pi}/2$. □

Exercise 15.4.41 (continued)

Exercise 15.4.41. *Converting to Polar Integrals.*
(b) Evaluate

$$\lim_{x\to\infty} \mathrm{ef}(x) = \lim_{x\to\infty} \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}}\, dt.$$

Solution. We have

$$\begin{aligned}\lim_{x\to\infty} \mathrm{erf}(x) &= \lim_{x\to\infty}\left(\int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}}\, dt\right) = \frac{2}{\sqrt{\pi}}\left(\lim_{x\to\infty}\int_0^x e^{-t^2}\, dt\right)\\ &= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{2}\right) \text{ by part (a)}\\ &= 1. \quad \square\end{aligned}$$

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