Chapter 13. Vector-Valued Functions and Motion in Space

13.4. Curvature and Normal Vectors of a Curve

Note. The rate at which $\mathbf{T}$ turns per unit of length along the curve is called the *curvature*. The symbol for curvature is kappa, κ. When a particle is traveling though space (or in a plane), then it can undergo an acceleration in two distinct ways: (1) an acceleration in the direction of travel, and (2) an acceleration which changes direction. This first type of acceleration does not reflect the shape of the curve, but how the curve is parametrized. By parametrizing the curve in terms of arc length, this type of acceleration becomes zero. Hence all of the acceleration is reflected in the second type of acceleration—the acceleration which results in a change of direction of the curve.

Definition. If $\mathbf{T}$ is the unit vector of a smooth curve, the *curvature* function of the curve is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|.$$

Note. If $|d\mathbf{T}/ds|$ is large, $\mathbf{T}$ turns sharply as the particle passes through P, and the curvature at P is large. If $|d\mathbf{T}/ds|$ is close to zero, $\mathbf{T}$ turns more slowly and the curvature at P is smaller. As we will see, the curvature of a circle of radius r is $\kappa = 1/r$.

Note. If a smooth curve $\mathbf{r}(t)$ is already given in terms of some parameter t other than the arc length s, we can calculate curvature using the Chain Rule:

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}}{dt}\frac{dt}{ds}\right| = \frac{1}{|ds/dt|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|.$$

Example. Page 747, Example 2. Consider the circle of radius a: $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$. Show that the curvature is $\kappa = 1/a$.

Definition. At a point where $\kappa \neq 0$, the *principal unit normal vector* for a smooth curve in the plane is

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$

Note. The vector $d\mathbf{T}/ds$ points in the direction in which $\mathbf{T}$ turns as the curve bends.

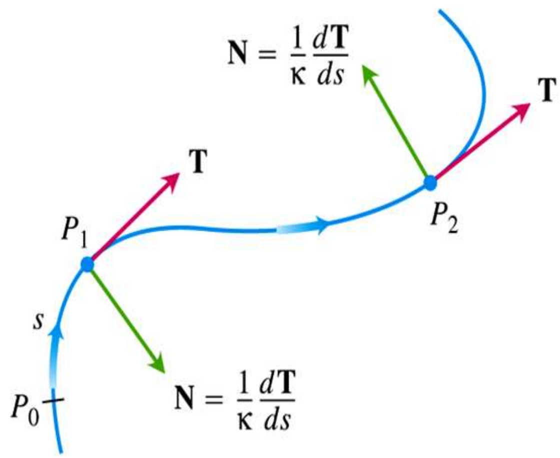

Figure 13.19, page 748

Note. If $\mathbf{r}(t)$ is not parametrized in terms of s, we can use the Chain Rule to calculate $\mathbf{N}$ in terms of t:

$$\mathbf{N} = \frac{d\mathbf{T}/ds}{\kappa} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{(d\mathbf{T}/dt)(dt/ds)}{|(d\mathbf{T}/dt)(dt/ds)|}$$

$$= \frac{(d\mathbf{T}/dt)(dt/ds)}{|(d\mathbf{T}/dt)||(dt/ds)|} = \frac{(d\mathbf{T}/dt)}{|(d\mathbf{T}/dt)} \text{ (since } dt/ds > 0).$$

Example. Page 751, number 2.

Definition. The *circle of curvature* or *osculating circle* at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

1. is tangent to the curve at P (has the same tangent line the curve has),

2. has the same curvature the curve has at P, and

3. lies toward the concave or inner side of the curve.

The *radius of curvature* of the curve at P is the radius of the circle of curvature, denoted ρ. The *center of curvature* of the curve at P is the center of the circle of curvature.

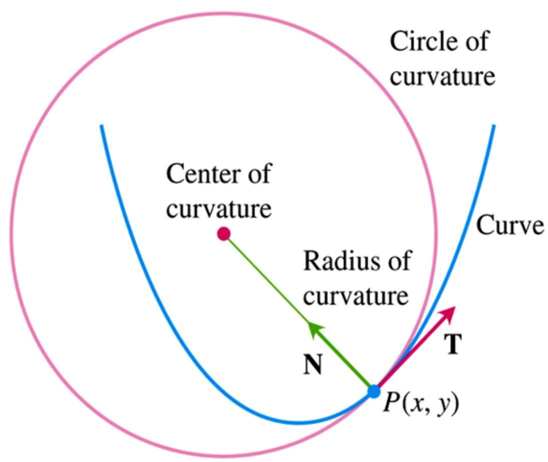

Figure 13.20, page 749

Example. Page 749, Example 4. Find and graph the osculating circle of the parabola $y = x^2$ at the origin. The solution is:

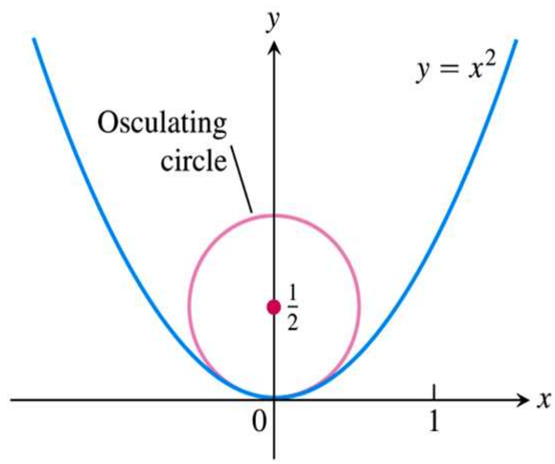

Figure 13.21, page 750

Note. We use the same formulae above, even for curves in three dimensions, instead of two.

Example. Page 751, number 12.